

DYNAMIC RESPONSE ANALYSIS OF COMPLEX
MECHANISMS WITH MULTIPLE INPUTS

By

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A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA IN PARTIAL
FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1971



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To Patricia

ACKNOWLEDGMENTS

The author expresses his appreciation to Dr. Delbert Tesar for his interest, encouragement, and assistance in all phases of his doctoral program as chairman of the supervisory committee, particularly in the preparation of this dissertation. Through Dr. Tesar's efforts and support the author was afforded an opportunity to present a major part of this research to the leading researchers in this area at the NSF Advanced Training Workshop in Mechanisms held at Oklahoma State University.

Sincere appreciation is expressed to Dr. Calvin C. Oliver for his assistance as co-chairman of the supervisory committee.

Appreciation is also expressed to the other committee members for their guidance and support:

Dr. W. H. Boykin, Jr.,

Dr. T. E. Bullock,

Dr. J. Mahig,

Dr. J. M. Vance.

The author is indebted to the Graduate Faculty for their support in making it possible for the author to receive financial assistance from a NDEA Title IV Fellowship.

Deep gratitude goes to the author's wife, Patricia, and daughter, Sharla, for their patience and understanding.

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NOMENCLATURE

$B_{m,n}$	- General linkage pin joint or points in links m,n
$\bar{C}_l$	- Viscous damping coefficient of dashpot attached to link l and ground
C_{ij}^*	- Equivalent viscous damping coefficient associated with the i^{th} input link due to a unit angular velocity of j^{th} input
E	- Coordinate point denoting center of gravity of system link
F	- General output coordinate point
$\bar{F}_e$	- External force acting through general system point E
$F_{e/i}^*$	- Equivalent force acting on i^{th} system input due to a unit external force at point E
l^{G_i}	- Velocity influence coefficient of link l with respect to input link i
l^{G_i}	- Row vector of velocity influence coefficients
$l^{h_{ij}}$	- Acceleration influence coefficient of link l with respect to the i^{th} and j^{th} input links
$l^{H_{ij}}$	- Square matrix of acceleration influence coefficients
i	- Denotes input link or generalized coordinate
$\bar{I}_l$	- Effective moment of inertia of link l taken about its center of gravity
I_{ij}^*	- Equivalent moment of inertia term

- j - Denotes input link counter
- k - Corresponding time counter
- k - Corresponding position counter
- $\bar{K}_\ell$ - Effective spring constant associated with a spring attached between link ℓ and ground
- $K_{\ell/i}^*$ - Equivalent spring constant of $\bar{K}_\ell$ with respect to system input i
- ℓ - Denotes general system link
- L^* - Lagrangian of the equivalent mass system
- m - Total number of general system outputs
- $\bar{M}_\ell$ - Effective mass of link ℓ at center of gravity
- n - Denotes total number of system inputs
- N - Denotes total number of system links
- P^i - Total system inertia power
- $ij^{P^*}_r$ - ij^{th} equivalent inertia power coefficient with respect to the r^{th} input
- q - Denotes input link counter
- r - Denotes input link counter
- S_1 - Summation representing the time integral of the Lagrangian
- S_2 - Summation representing the time integral of the virtual work of the nonconservative forces
- $\bar{T}_\ell$ - General system external torque acting on link ℓ
- $T_{\ell,e,d,s/i}^*$ - Equivalent torque acting at i^{th} input link resulting from unit torques on link ℓ , unit forces through system point E, unit velocities on equivalent viscous damper, and unit displacements on equivalent springs

T_i^*	- Total equivalent torques acting on i^{th} system input link
v_e	- Linear velocity of general system point E
W_l	- Weight of link l acting through center of gravity
x	- Denotes x-coordinate of system input
X	- Denotes x-coordinate of system output
y	- Denotes y-coordinate of system input
Y	- Denotes y-coordinate of system output
α_i	- Angular acceleration of link i
δ	- Variation of some parameter
Δ	- Finite increment
η	- Denotes angles
λ	- Undetermined Lagrangian multiplier
φ_i	- Angle of i^{th} system input link
ϕ_l	- Angle of l^{th} system output link
ω_i	- Angular velocity of link i
$ \quad $	- Denotes column vector
$\cdot$	- Denotes differentiation with respect to time
$'$	- Denotes transpose of matrix
∂	- Denotes partial differentiation

Sign Convention

Right-hand Cartesian coordinate system

Angles measured positive ccw from positive x-axis

Abstract of Dissertation Presented to the
Graduate Council of the University of Florida in Partial Fulfillment
of the Requirements for the Degree of Doctor of Philosophy

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December, 1971

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Co-Chairman: Dr. Calvin C. Oliver
Major Department: Mechanical Engineering

The holonomic constraints associated with complex, multiple input linkage systems complicate the procedures and methods used in determining their dynamic response. Large systems of nonlinear, second-order differential equations, requiring additional algebraic equations of constraint, occur as a result of these constraints. Double iteration algorithms, which are both time-consuming and subject to error, are necessary to integrate numerically these differential equations of motion.

In this dissertation the concepts of kinematic influence coefficients of complex, planar, rigid link mechanisms with multiple inputs are developed and utilized to eliminate the holonomic constraints associated with such systems. Kinematic influence coefficients associated with series and parallel linkage combinations are developed, based on the addition of Assur groups (dyads, tetrads and more complex groups) to the basic system group.

These complex, multiple input linkage systems are then reduced to coupled equivalent mass systems acted upon by variable rate springs, variable coefficient viscous dampers, and equivalent external forces and torques. The holonomic constraints associated with the original system are eliminated, thus leaving the equivalent mass system free of all such constraints. The number of generalized coordinates required to describe the motion of the equivalent system now equals the number of independent system inputs.

The differential equations of motion describing the system's dynamical behavior can then be determined by established methods and put in a suitable form for numerical integration.

CHAPTER I

LITERATURE SURVEY

A substantial survey of the literature representing the state of the art in dynamic response analysis of constrained mechanisms appears in [1]. Virtually all systems treated in the literature possess a single degree of freedom. Furthermore, these systems are analyzed for their output dynamic state. Few researchers have attempted to determine the input dynamic state of complex mechanisms with multiple inputs (degrees of freedom) to forces and torques. Those who have, invariably give examples which are constrained four-link mechanisms, thus leaving their methods essentially untested. This emphasizes the need for a formalized procedure to model complex systems mathematically, utilizing direct methods for determining their dynamic response to known forces, torques and energy crossing the system boundaries. This will eliminate iterative procedures necessary in solving the differential equations of motion.

Dynamic State Analysis

As stated in [1] "dynamic state" implies that the velocity and acceleration of each point of every link of the system are piecewise continuous, differentiable functions of the input characteristics. In 1957 Modrey [2] developed a graphical method whereby the velocities

and accelerations of the links of a system of higher-order-complexity (one which cannot be analyzed as a system of four-link mechanisms connected in series) could be determined, using velocity and acceleration influence coefficients obtained through a "zero-relax" procedure. In a discussion of this paper, T. P. Goodman showed the analytical equivalent to Modrey's method and proved the linearity property necessary for superposition with respect to single degree of freedom systems. Other graphical methods existed for solving the dynamic state question of complex mechanisms; namely, Hall and Alt's method, Carter's method and the method of normal accelerations. All of these methods assume a known geometric configuration. That is, the point path is known, and the dynamic state is determined based on this known configuration.

Displacement Analysis

Various methods have been developed for the displacement analysis of mechanisms containing more than one vector loop. Hain [3] shows how to determine the input-output position information of six- and eight-bar mechanisms which do not contain a basic four-bar at ground but do contain an internal four-link loop. This is accomplished by inverting the mechanism, solving the position information and then reinverting. However, most methods rely on harmonic analysis, including those by Meyer zur Capellen [4], Denavit and Hasson [5], Flory and Wolford [6], Markus and Tomas [7], and the Romanian school [8,9]. They obtain the harmonics by approximate numerical means within a prescribed error criterion. Crossley and Seshachar [10] analyze the displacement of planar Assur

groups by an iterative algebraic method which allows half the unknowns from the matrix of complex numbers to be found first with the other half found later. All of these methods rely on iteration or approximations because of the complex loop equations resulting from the geometry. However, they yield results which can be made as accurate as desired. Digital computation capability increases the desirability of these methods.

Dynamic Response Analysis

Time response of mechanical systems, spatial as well as planar, by use of Lagrangian mechanics has been advanced during the most recent years. Chace [11] uses relative coordinates in determining the dynamic response of multiple degree of freedom spatial mechanisms. He utilizes Lagrangian multipliers to account for the physical (geometric) constraints. Smith [12] employs this same technique in analyzing the reaction forces in generalized machine systems. In both [11] and [12] the examples are planar single degree of freedom four-link mechanisms, leaving the general nature of the method untested.

Uicker [13] and Carson and Trummel [14] apply matrix notation and methods to the kinematics problem. By applying Lagrangian techniques, they develop the system differential equations based on the vector loop displacement equations. This method has been the most useful and powerful for analyzing the dynamic response of complex multiple degree of freedom systems to date. As with other methods, it has its weaknesses. A dual iteration algorithm results, which is ideal for digital computation but is time-consuming.

The works of Wittenbauer [15], Federhofer [16], and Beyer [17] are significant in the development of equivalent mass systems and appear to be the most direct antecedents of the concepts developed in this dissertation. Wittenbauer and Federhofer introduce the concepts of reduced mass in terms of general mass content and velocity ratios. Beyer effectively summarized this work for the modern reader in terms of the time response problem. The authors in references [11-14] ignore this property, preferring to treat these systems as ones with large numbers of generalized coordinates, coupled by algebraic equations of constraint. This property necessitates the use of the Lagrangian λ -method in treating the holonomic auxiliary conditions, replacing the kinematical constraints by the forces necessary in maintaining them. The λ 's, then, indicate the degree to which the constraints are violated.

The need for a more concise system formulation, allowing one to determine the dynamic response in a more precise and direct manner, is paramount. This will be accomplished in the following chapters by developing the concepts of kinematic influence coefficients of complex, planar, rigid-link mechanisms with multiple inputs (degrees of freedom). These, in effect, account for the holonomic constraints on the system, resulting in coupled, equivalent mass systems requiring only as many generalized coordinates as its degrees of freedom. These equivalent systems can then be uncoupled by standard methods, allowing their time response to be determined by direct methods such as that developed in [18].

CHAPTER II

KINEMATIC INFLUENCE COEFFICIENTS OF COMPLEX MECHANISMS CONSTRUCTED FROM ASSUR GROUPS

Pelecudi [19,20] defines kinematic multipoles as kinematic chains interpreted as lines for transmitting motion information (position, velocity, acceleration) of rigid links from an input to an output or, more generally, from one point in the chain to another. This is accomplished by motion transforms called kinematic ratios, velocity and acceleration ratios, or, more precisely, kinematic influence coefficients. These are well established for the case of four-link mechanisms and cams in [1,21-25]. Kinematic chains receive information from input poles and transmit it to output poles. The generality of this concept enables one to construct highly complex mechanisms consisting of groups of links known as Assur groups. The degree of mobility of the final system can therefore be controlled or determined by the mobility associated with each group added to the basic kinematic chain. Two types of chains are most important in the construction of complex mechanisms.

(1) Structural groups having a degree of mobility (freedom)

$f \leq 0$, forming *passive* multipoles. A group may be overconstrained ($f < 0$) to form a mechanism with a degree of mobility which is less than the basic chain.

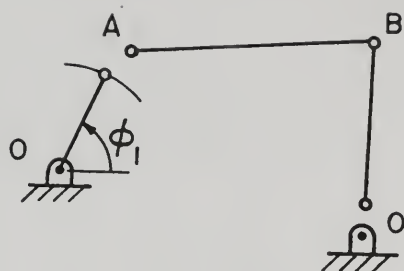
- (2) Mechanisms with a fixed link having a degree of freedom $f \geq 1$, forming *active* multipoles.

Thus, one is free to construct mechanisms from basic chains whose degree of freedom remains invariant under the addition of *active* and *passive* multipoles. This points out the importance of the structural group in the construction of mechanisms of high complexity.

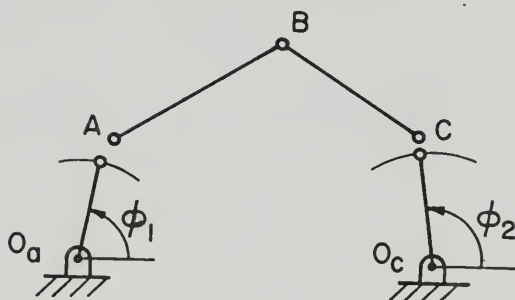
For the purpose of analysis it is desirable to reduce these systems to their basic kinematic groups in order to derive the input-output transformation of position, velocity, and acceleration. Systems possessing these transformations can be reduced to coupled equivalent mass systems. This coupling will be quadratic and will depend on velocity influence coefficients.

It is necessary to discuss the types of kinematic groups available for constructing complex mechanisms of desired mobility.

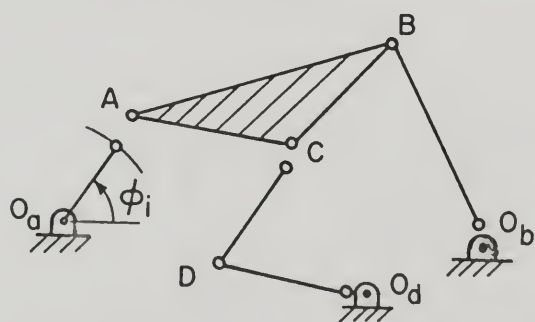
Assur groups [26] are defined in terms of kinematic groups which form structures when the outer joints are fixed after being separated from the input links. The binary group is the simplest of the Assur groups. Figure 2-1 illustrates its use in constructing the four-, five-, and six-bar mechanisms. Each binary group forms a structure when it is separated from its primary or intermediate input and has its outer joints fixed. The dyad in the four- and five-bar mechanisms and the dyad attached to the intermediate input of the six-bar mechanism form dipoles while the dyad in Figure 2-1(c) attached to the primary input forms a tripole, capable of transmitting information from the system input through the output pole to the dipole.



(a)



(b)



(c)

Figure 2-1. Binary Groups

The next Assur group of higher complexity is the tetrad (see Figure 2-2(a)). When the outer joints A, O_c , and E are fixed, the group becomes a structure with $f = 0$. This group requires two vector loop equations to determine the angular relationships between the links. More complex Assur groups are illustrated in Figures 2-2(b) and (c). These and other groups are classified and analyzed for their displacements by Crossley and Seshachar [10].

Markus and Tomas [7] define a system group as a group of links which has zero degrees of freedom when the input links are fixed. The system groups contain Assur groups but are more general, enabling the authors to solve for displacement information of any system group by a single harmonic analysis algorithm. Curtis and Tomas [27] point out the importance of the binary group in determining position information of mechanisms of varying complexity. They utilize the law of cosines in developing a closed form algorithm to obtain the position information of a large class of mechanisms consisting of various binary group configurations.

Kinematic Influence Coefficients of Primary Input Poles

Velocity Influence Coefficients

An implicit functional relationship, $f(x,y) = 0$, describing the path of some general system point E (see Figure 2-3(a)) associated with the outer joint of a binary group, can be written based on known position information. In general, $\bar{g}_e$, $\bar{h}_e$, m_e , φ_l , and φ_{l+1} will be known as a function of some system input parameter such as φ_1 . It is

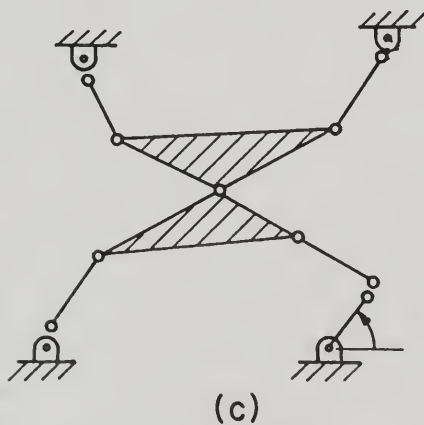
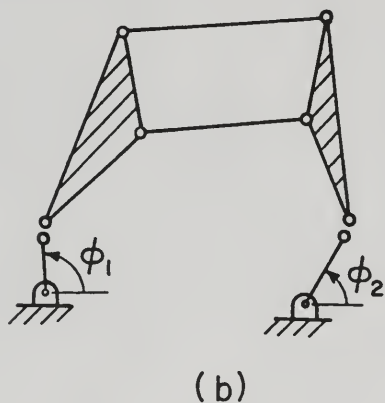
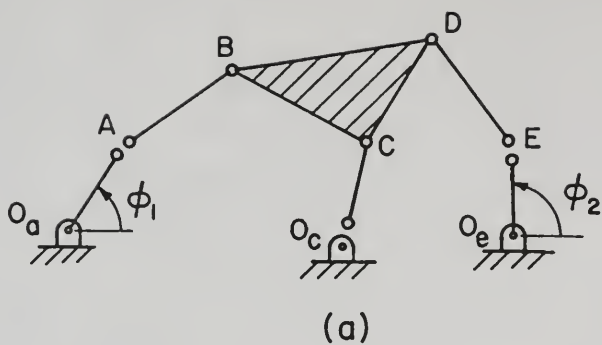
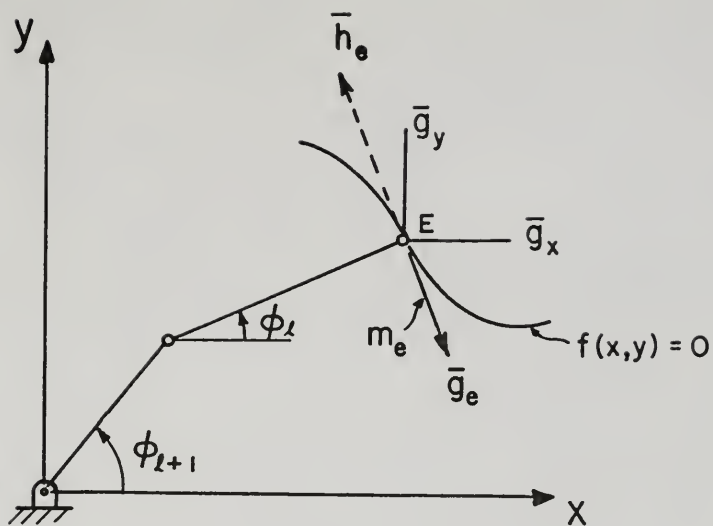
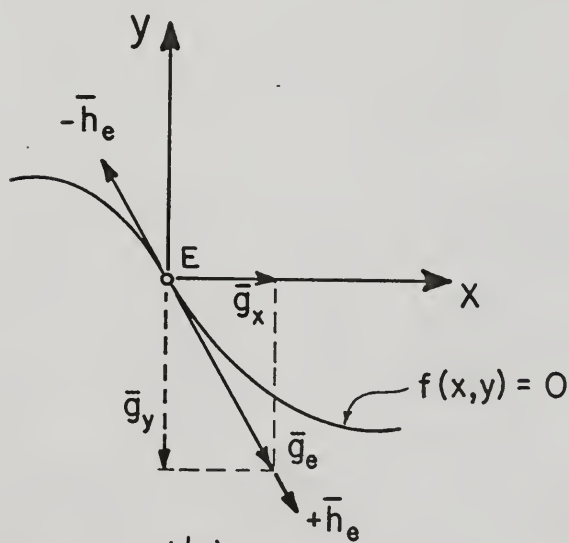


Figure 2-2. Assur Groups



(a)



(b)

Figure 2-3. General System Point

noted that $\bar{g}_e$ and $\bar{h}_e$ are defined in [1] as

$$\bar{g}_e \equiv \frac{dS_e}{d\varphi_i},$$

$$\bar{h}_e \equiv \frac{d^2 S_e}{d\varphi_i^2},$$

and

m_e = slope of path tangent of point S_e .

In order to proceed through the chain and determine the influence coefficients associated with $F(X,Y)$ the values for $\bar{g}_x, \bar{g}_y, \bar{h}_x, \bar{h}_y$ must be determined. It is noted that (see Figure 2-4(a))

$$v_x = \bar{g}_x \dot{\varphi}_i = \frac{dS_{ex}}{d\varphi_i} \dot{\varphi}_i, \quad (2-1)$$

$$v_y = \bar{g}_y \dot{\varphi}_i = \frac{dS_{ey}}{d\varphi_i} \dot{\varphi}_i,$$

and

$$\begin{aligned} \frac{d}{dt} \{f(x,y)\} &= \frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y \\ &= \frac{\partial f}{\partial x} \bar{g}_x + \frac{\partial f}{\partial y} \bar{g}_y = 0. \end{aligned} \quad (2-2)$$

This gives

$$\frac{\bar{g}_y}{\bar{g}_x} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \quad (2-3)$$

The expressions for m_e and $\bar{g}_e$ are given by

$$m_e = \frac{\bar{g}_y}{\bar{g}_x}, \quad (2-4)$$

and

$$\bar{g}_e = \sqrt{\bar{g}_x^2 + \bar{g}_y^2}. \quad (2-5)$$

Since $\bar{g}_e$ is known, then

$$\bar{g}_x = \frac{\bar{g}_c}{\sqrt{1 + m_e^2}}, \quad (2-6)$$

and

$$\bar{g}_y = m_e \bar{g}_x \quad (2-7)$$

Acceleration Influence Coefficients

Now consider $\bar{h}_x$ and $\bar{h}_y$ in terms of m_e , $\bar{g}_e$ and $\bar{h}_e$.

It is noted that

$$\begin{aligned} \bar{h}_x &= a_x^t, \\ \bar{h}_y &= a_y^t, \text{ for } \dot{\phi}_i = 1.0, \quad \ddot{\phi}_i = 0. \end{aligned} \quad (2-8)$$

Define η as follows (see Figure 2-3(b))

$$\eta = \tan^{-1} \left\{ \frac{\bar{g}_y}{\bar{g}_x} \right\}. \quad (2-9)$$

For $f(x,y) = 0$, a closed path of point E in a general system link,

$\bar{g}_e$ is always positive. Then $\bar{h}_e$ is positive if directed in the same sense as $\bar{g}_e$, negative if directed in the opposite direction.

Therefore,

$$\begin{aligned}\bar{h}_x &= \bar{h}_e \cos (\eta) , \\ \bar{h}_y &= \bar{h}_e \sin (\eta) .\end{aligned}\tag{2-10}$$

Now consider a dyad as a dipole or binary group in the Assurian sense with points E_1 and E_2 having general point paths defined by implicit functions $f_1(x_1, y_1) = 0$ and $f_2(x_2, y_2) = 0$ (see Figure 2-4(a)). Then,

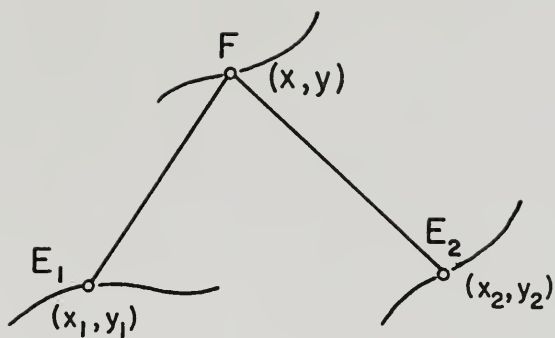
$$\begin{aligned}\bar{g}_{x_1} &= \frac{\bar{g}_{e_1}}{\sqrt{1 + m_{e_1}^2}} , & \bar{g}_{y_1} &= m_{e_1} \bar{g}_{x_1} , \\ \bar{g}_{x_2} &= \frac{\bar{g}_{e_2}}{\sqrt{1 + m_{e_2}^2}} , & \bar{g}_{y_2} &= m_{e_2} \bar{g}_{x_2} ,\end{aligned}\tag{2-11}$$

and

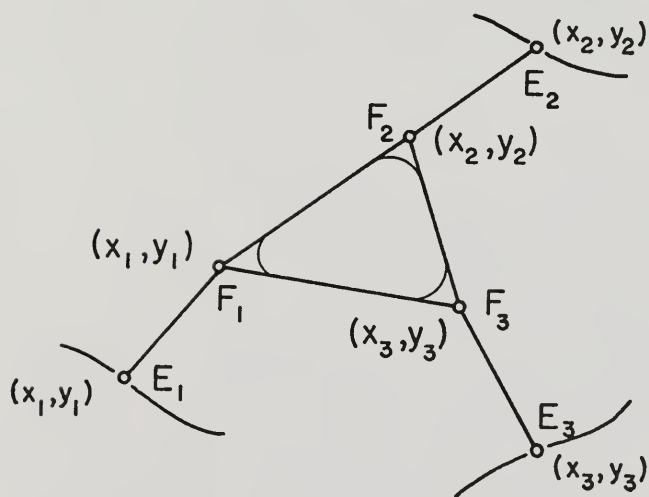
$$\begin{aligned}\bar{h}_{x_1} &= \bar{h}_{e_1} \cos (\eta_1) , & \bar{h}_{y_1} &= \bar{h}_{e_1} \sin (\eta_1) , \\ \bar{h}_{x_2} &= \bar{h}_{e_2} \cos (\eta_2) , & \bar{h}_{y_2} &= \bar{h}_{e_2} \sin (\eta_2) .\end{aligned}\tag{2-12}$$

This can be generalized immediately to all classes of Assur groups (see Figures 2-4(a) and (b)) as

$$\begin{aligned}\bar{g}_{x_i} &= \frac{\bar{g}_{e_i}}{\sqrt{1 + m_{e_i}^2}} , & \bar{g}_{y_i} &= m_{e_i} \bar{g}_{x_i} , \\ \bar{h}_{x_i} &= \bar{h}_{e_i} \cos (\eta_i) , & \bar{h}_{y_i} &= \bar{h}_{e_i} \sin (\eta_i) .\end{aligned}\tag{2-13}$$



(a)



(b)

Figure 2-4. Point Paths of Assur Groups

Kinematic Influence Coefficients
of Intermediate Input Poles

Let E_1 and E_2 be the inputs and F be the output of the binary group in Figure 2-4(a). Point F is an internal joint to links l and $l+1$. However, the following derivation holds for output poles as general points in either of the links l or $l+1$. The point path of F can be expressed as an implicit function

$$f(X,Y) = 0, \quad (2-14)$$

where X and Y are functions of x_1, y_1, x_2 , and y_2 . Relation (2-14) may be expressed by the projections of the vector loop equation for the dyad,

$$\overline{E_1 F} + \overline{F E_2} + \overline{E_2 E_1} = 0, \quad (2-15)$$

as

$$f_x(x_1, x_2, X, y_1, y_2, Y) = 0,$$

and

$$f_y(x_1, x_2, X, y_1, y_2, Y) = 0. \quad (2-16)$$

Velocity Influence Coefficients

Since X, Y are functions of x_1, y_1, x_2 , and y_2 , then they are differentiable with respect to them. The velocity of point F is given

as

$$\vec{V}_F = \vec{v}_X + \vec{v}_Y \quad (2-17)$$

where

$$v_X = \sum_{i=1}^2 \left\{ \frac{\partial X}{\partial x_i} v_{x_i} + \frac{\partial X}{\partial y_i} v_{y_i} \right\},$$

$$v_Y = \sum_{i=1}^2 \left\{ \frac{\partial Y}{\partial x_i} v_{x_i} + \frac{\partial Y}{\partial y_i} v_{y_i} \right\}.$$
(2-18)

The partial derivatives in Equations (2-18) represent the velocity influence coefficients for multiple input linkage systems. They are defined as follows in order to remain consistent with previous literature.

Let

$$X^G_{x_i} \triangleq \frac{\partial X}{\partial x_i},$$

$$Y^G_{y_i} \triangleq \frac{\partial Y}{\partial y_i}.$$
(2-19)

Substituting these into Equations (2-18) gives

$$v_X = \sum_{i=1}^2 \left\{ X^G_{x_i} v_{x_i} + X^G_{y_i} v_{y_i} \right\},$$

$$v_Y = \sum_{i=1}^2 \left\{ Y^G_{x_i} v_{x_i} + Y^G_{y_i} v_{y_i} \right\}.$$
(2-20)

Equations (2-20) can be put into compound matrix form as

$$\begin{bmatrix} v_X \\ v_Y \end{bmatrix} = \begin{bmatrix} X^G_{x_i} & X^G_{y_i} \\ Y^G_{x_i} & Y^G_{y_i} \end{bmatrix} \begin{bmatrix} |v_x| \\ |v_y| \end{bmatrix}.$$
(2-21)

Similarly for angular output information

$$\begin{bmatrix} \omega_{\ell} \end{bmatrix} = \begin{bmatrix} \ell^{G_{x_i}} & \vdots & \ell^{G_{y_i}} \end{bmatrix} \begin{bmatrix} |v_{x_i}| \\ \vdots \\ |v_{y_i}| \end{bmatrix}, \quad (2-22)$$

where

$$\begin{aligned} \bar{x}^{G_{x_i}} &\equiv \begin{bmatrix} \frac{\partial x}{\partial x_1} & \frac{\partial x}{\partial x_2} \end{bmatrix}, \\ \ell^{G_{x_i}} &\equiv \begin{bmatrix} \frac{\partial \bar{\ell}}{\partial x_1} & \frac{\partial \bar{\ell}}{\partial x_2} \end{bmatrix}, \\ |v_x| &\equiv \begin{bmatrix} v_{x_1} \\ v_{x_2} \end{bmatrix}. \end{aligned} \quad (2-23)$$

Acceleration Influence Coefficients

Differentiating Equations (2-18) with respect to time yields

$$\begin{aligned} a_X^t &= \sum_{i=1}^2 \sum_{j=1}^2 \left\{ \frac{\partial^2 x}{\partial x_i \partial x_j} v_{x_i} v_{x_j} + 2 \frac{\partial^2 x}{\partial x_i \partial y_j} v_{x_i} v_{y_j} \right. \\ &\quad \left. + \frac{\partial^2 x}{\partial y_i \partial y_j} v_{y_i} v_{y_j} \right\} + \sum_{i=1}^2 \left\{ \frac{\partial x}{\partial x_i} a_{x_i}^t + \frac{\partial x}{\partial y_i} a_{y_i}^t \right\}, \end{aligned} \quad (2-24)$$

and

$$\begin{aligned} a_Y^t &= \sum_{i=1}^2 \sum_{j=1}^2 \left\{ \frac{\partial^2 y}{\partial x_i \partial x_j} v_{x_i} v_{x_j} + 2 \frac{\partial^2 y}{\partial x_i \partial y_j} v_{x_i} v_{y_j} \right. \\ &\quad \left. + \frac{\partial^2 y}{\partial y_i \partial y_j} v_{y_i} v_{y_j} \right\} + \sum_{i=1}^2 \left\{ \frac{\partial y}{\partial x_i} a_{x_i}^t + \frac{\partial y}{\partial y_i} a_{y_i}^t \right\}. \end{aligned} \quad (2-25)$$

Equations (2-24) and (2-25) can be written in compound matrix form and generalized to n inputs as

$$\begin{aligned} a_X^t = & |v_x|^t \left[\bar{X}_{x_i x_j} \right] |v_x| + 2 |v_x|^t \left[\bar{X}_{x_i y_j} \right] |v_y| \\ & + |v_y|^t \left[\bar{X}_{y_i y_j} \right] |v_y| + \left[\bar{X}_{x_i} \mid \bar{X}_{y_i} \right] \begin{bmatrix} |a_x^t| \\ \hline |a_y^t| \end{bmatrix}, \quad (2-26) \end{aligned}$$

$$\begin{aligned} a_Y^t = & |v_x|^t \left[\bar{Y}_{x_i x_j} \right] |v_x| + 2 |v_x|^t \left[\bar{Y}_{x_i y_j} \right] |v_y| \\ & + |v_y|^t \left[\bar{Y}_{y_i y_j} \right] |v_y| + \left[\bar{Y}_{x_i} \mid \bar{Y}_{y_i} \right] \begin{bmatrix} |a_x^t| \\ \hline |a_y^t| \end{bmatrix}, \quad (2-27) \end{aligned}$$

$$i, j = 1, 2, \dots, n,$$

where

$$\left[\bar{X}_{x_i x_j} \right] \equiv \begin{bmatrix} \bar{X}_{x_1 x_1} & \bar{X}_{x_1 x_2} & \cdots & \bar{X}_{x_1 x_n} \\ \bar{X}_{x_2 x_1} & \bar{X}_{x_2 x_2} & \cdots & \bar{X}_{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{X}_{x_n x_1} & \bar{X}_{x_n x_2} & \cdots & \bar{X}_{x_n x_n} \end{bmatrix}, \quad (2-28)$$

$$\bar{X}_{x_i x_j} \equiv \frac{\partial^2 X}{\partial x_i \partial x_j}, \quad (2-29)$$

and

$$\left[\bar{X}_{x_i} \mid \bar{X}_{y_i} \right] \equiv \left[\bar{X}_{x_1} \cdots \bar{X}_{x_n} \mid \bar{X}_{y_1} \cdots \bar{X}_{y_n} \right]. \quad (2-30)$$

Corresponding expressions relating angular input to angular or linear output can be derived. Since most kinematic chains consist of lower pair connections, the remaining derivations will be based on angular inputs to both linear and angular outputs. Complex kinematic chains constructed from basic chains with invariant mobility are developed by adding Assur groups to output poles of the established chain of predetermined mobility. If the mobility does not remain invariant, then the basic kinematic chain with its known input-output position relationships is destroyed and the point paths are no longer predetermined with respect to the original system group inputs. A new system group must be established and new input-output relationships developed.

Total Input-Output Kinematic Influence Coefficients

If the inputs to a linkage system group are angular, then the intermediate linear inputs (x_i, y_i) and their dynamic states (v_{x_i}, v_{y_i}) and $(a_{x_i}^t, a_{y_i}^t)$ can be expressed as functions of the input position parameters φ_q 's and their dynamic states ω_q 's and α_q 's. The derivation of the total input-output transformation is now carried out.

Velocity Influence Coefficients

Consider the system group in Figure 2-5. The expression for the velocity components of point F with respect to points E_1 and E_2 are

$$\begin{bmatrix} v_X \\ v_Y \end{bmatrix} = \begin{bmatrix} \bar{x} \bar{G}_{x_i} & \bar{x} \bar{G}_{y_i} \\ \bar{y} \bar{G}_{x_i} & \bar{y} \bar{G}_{y_i} \end{bmatrix} \begin{bmatrix} |v_x| \\ |v_y| \end{bmatrix} \quad (2-31)$$

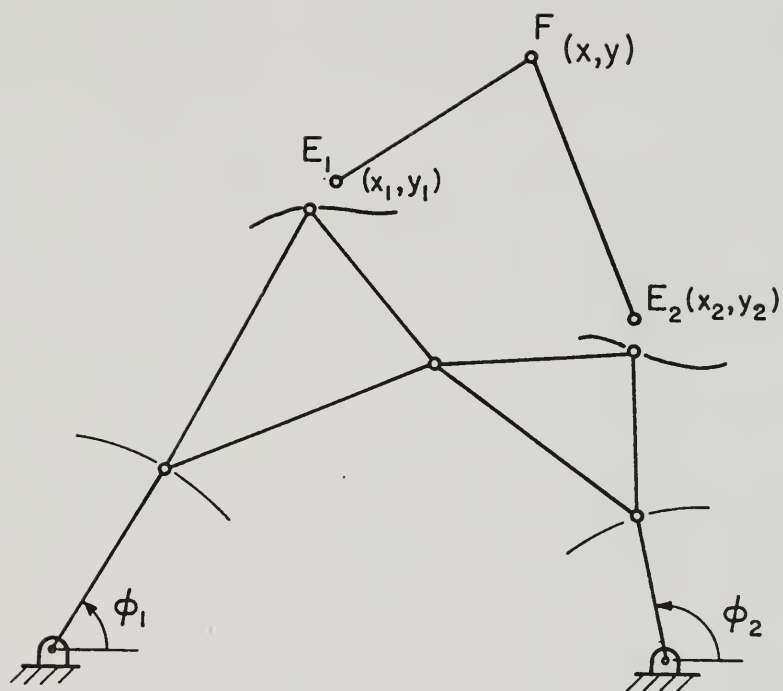


Figure 2-5. Seven-Link System Group

The intermediate input data, in terms of the primary input information, are given by

$$\begin{bmatrix} |v_x| \\ \hline |v_y| \end{bmatrix} = \begin{bmatrix} x_i \bar{G}_{\varphi_q} \\ y_i \bar{G}_{\varphi_q} \end{bmatrix} |\omega|, \quad i, q = 1, 2, \quad (2-32)$$

where

$$\begin{bmatrix} x_i \bar{G}_{\varphi_q} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial \varphi_1} & \frac{\partial x_1}{\partial \varphi_2} \\ \frac{\partial x_2}{\partial \varphi_1} & \frac{\partial x_2}{\partial \varphi_2} \end{bmatrix},$$

and

$$|\omega| = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}.$$

Substituting the right-hand side of Equation (2-32) into Equation (2-31) yields

$$\begin{bmatrix} v_X \\ v_Y \end{bmatrix} = \begin{bmatrix} X \bar{G}_{x_i} & X \bar{G}_{y_i} \\ Y \bar{G}_{x_i} & Y \bar{G}_{y_i} \end{bmatrix} \begin{bmatrix} x_i \bar{G}_{\varphi_q} \\ y_i \bar{G}_{\varphi_q} \end{bmatrix} |\omega|, \quad (2-33)$$

which reduces to

$$\begin{bmatrix} v_X \\ v_Y \end{bmatrix} = \begin{bmatrix} X \bar{G}_{\varphi_q} \\ Y \bar{G}_{\varphi_q} \end{bmatrix} |\omega|, \quad q = 1, 2. \quad (2-34)$$

Note that when $q = 1$, Equation (2-34) becomes

$$\begin{aligned} v_X &= \bar{X}_{11}^G \omega_1, \\ v_Y &= \bar{Y}_{11}^G \omega_1, \end{aligned} \quad (2-35)$$

which are recognized as the velocity influence coefficients developed in reference [1] for single input systems.

Acceleration Influence Coefficients

The component accelerations of point F (Figure 2-5) in terms of the intermediate inputs are

$$\begin{aligned} \begin{bmatrix} a_X^t \\ a_Y^t \end{bmatrix} &= \begin{bmatrix} |v_X|' & 0 \\ 0 & |v_X|' \end{bmatrix} \begin{bmatrix} \bar{X}_{x_i x_j}^H \\ \bar{Y}_{x_i x_j}^H \end{bmatrix} |v_X| \\ &+ 2 \begin{bmatrix} |v_X|' & 0 \\ 0 & |v_X|' \end{bmatrix} \begin{bmatrix} \bar{X}_{x_i y_j}^H \\ \bar{Y}_{x_i y_j}^H \end{bmatrix} |v_Y| + \begin{bmatrix} |v_Y|' & 0 \\ 0 & |v_Y|' \end{bmatrix} \begin{bmatrix} \bar{X}_{y_i y_j}^H \\ \bar{Y}_{y_i y_j}^H \end{bmatrix} |v_Y| \\ &+ \begin{bmatrix} \bar{X}_{x_i}^G & \bar{X}_{y_i}^G \\ \bar{Y}_{x_i}^G & \bar{Y}_{y_i}^G \end{bmatrix} \begin{bmatrix} |a_x^t| \\ |a_y^t| \end{bmatrix}. \end{aligned} \quad (2-36)$$

The first and third terms of Equation (2-36) are recognized as real quadratic forms while the second is real bilinear.

Substitution of Equations (2-26), (2-27), and (2-32) into

Equation (2-36) gives

$$\begin{aligned}
 & \begin{matrix} (2N \times 1) & (2N \times 2n) & (2n \times 2n) & (2n \times n) & (n \times n) & (n \times 1) \end{matrix} \\
 & \begin{bmatrix} a_X^t \\ a_Y^t \end{bmatrix} = \begin{bmatrix} |w|' & 0 \\ 0 & |w|' \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x_i \bar{G}_{\varphi_q} \end{bmatrix}' & 0 \\ 0 & \begin{bmatrix} x_i \bar{G}_{\varphi_q} \end{bmatrix}' \end{bmatrix} \begin{bmatrix} \bar{X}^H x_i x_j \\ \bar{Y}^H x_i x_j \end{bmatrix} \begin{bmatrix} x_j \bar{G}_{\varphi_q} \end{bmatrix} |w| \\
 & + 2 \begin{bmatrix} |w|' & 0 \\ 0 & |w|' \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x_i \bar{G}_{\varphi_q} \end{bmatrix}' & 0 \\ 0 & \begin{bmatrix} x_i \bar{G}_{\varphi_q} \end{bmatrix}' \end{bmatrix} \begin{bmatrix} \bar{X}^H x_i y_j \\ \bar{Y}^H x_i y_j \end{bmatrix} \begin{bmatrix} y_j \bar{G}_{\varphi_q} \end{bmatrix} |w| \\
 & + \begin{bmatrix} |w|' & 0 \\ 0 & |w|' \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x_i \bar{G}_{\varphi_q} \end{bmatrix}' & 0 \\ 0 & \begin{bmatrix} y_i \bar{G}_{\varphi_q} \end{bmatrix}' \end{bmatrix} \begin{bmatrix} \bar{X}^H y_i y_j \\ \bar{Y}^H y_i y_j \end{bmatrix} \begin{bmatrix} y_j \bar{G}_{\varphi_q} \end{bmatrix} |w| \\
 & \begin{matrix} (2N \times 2n) & (2n \times 2nn) & (2nn \times n) & (n \times 1) \end{matrix} \\
 & + \begin{bmatrix} \bar{X}^H x_i & \bar{X}^H y_i \\ \bar{Y}^H x_i & \bar{Y}^H y_i \end{bmatrix} \begin{bmatrix} |w|' & 0 & 0 & 0 \\ 0 & |w|' & 0 & 0 \\ 0 & 0 & |w|' & 0 \\ 0 & 0 & 0 & |w|' \end{bmatrix} \begin{bmatrix} x_i \bar{H}_{\varphi_q \varphi_r} \\ y_i \bar{H}_{\varphi_q \varphi_r} \end{bmatrix} |w| \\
 & \left. \begin{matrix} (2n \times n) & (n \times 1) \\ + \begin{bmatrix} x_i \bar{G}_{\varphi_q} \\ y_i \bar{G}_{\varphi_q} \end{bmatrix} \right\} \alpha \right\}, \quad N, n = 2. \quad (2-37)
 \end{aligned}$$

The complexity of Equation (2-37) is reduced by collecting like terms and writing it in a more concise and familiar form as

$$\begin{bmatrix} t \\ a_X \\ t \\ a_Y \end{bmatrix} = \left\| \omega \right\|' \begin{bmatrix} X \bar{H} \varphi_q \varphi_r \\ \text{-----} \\ Y \bar{H} \varphi_q \varphi_r \end{bmatrix} |\omega| + \begin{bmatrix} X \bar{G} \varphi_q \\ \text{-----} \\ Y \bar{G} \varphi_q \end{bmatrix} |\alpha|, \quad q, r = 1, 2, \quad (2-38)$$

where $\left\| \omega \right\|'$ is a compound diagonal matrix given by

$$\left\| \omega \right\|' = \left[\begin{array}{c|c} |\omega|' & 0 \\ \hline 0 & |\omega|' \end{array} \right], \quad (2-39)$$

and $|\omega|'$ is the transpose of the column vector $|\omega|$.

The expressions (2-34) and (2-38) for the velocity and acceleration can be generalized to treat systems with n inputs and m outputs.

These expressions take the general form

$$\begin{bmatrix} v_{X_1} \\ \vdots \\ v_{X_m} \\ v_{Y_1} \\ \vdots \\ v_{Y_m} \end{bmatrix} = \begin{bmatrix} X_1 \bar{G} \varphi_q \\ \text{-----} \\ X_m \bar{G} \varphi_q \\ Y_1 \bar{G} \varphi_q \\ \text{-----} \\ Y_m \bar{G} \varphi_q \end{bmatrix} |\omega|, \quad q = 1, 2, \dots, n, \quad (2-40)$$

$$\begin{bmatrix} a_{X_1}^t \\ \vdots \\ a_{X_m}^t \\ a_{Y_1}^t \\ \vdots \\ a_{Y_m}^t \end{bmatrix} = \| \omega \|' \begin{bmatrix} X_1 \bar{H}_q \varphi_r \\ \vdots \\ X_m \bar{H}_q \varphi_r \\ Y_1 \bar{H}_q \varphi_r \\ \vdots \\ Y_m \bar{H}_q \varphi_r \end{bmatrix} |\omega| + \begin{bmatrix} X_1 \bar{G}_q \\ \vdots \\ X_m \bar{G}_q \\ Y_1 \bar{G}_q \\ \vdots \\ Y_m \bar{G}_q \end{bmatrix} |\alpha|, \quad (2-41)$$

$$q, r = 1, \dots, n.$$

It is clear that expressions similar to Equations (2-40) and (2-41) can be written for the angular properties of the system links. For the purpose of illustration, both forms will be used where appropriate throughout the remainder of this dissertation.

The discrete form [28] of the kinematic influence coefficients of velocity and acceleration are expressed as

$$\begin{aligned} \ell_{ik}^g &= \left(\frac{\partial \bar{\phi}}{\partial \varphi_i} \right)_k, \\ \ell_{ik}^g &= \frac{\ell_{i,k+1}^{\bar{\phi}} - \ell_{i,k-1}^{\bar{\phi}}}{2\Delta\varphi_i}, \end{aligned} \quad (2-42)$$

$$\begin{aligned} \ell_{ij,k}^h &= \left(\frac{\partial^2 \bar{\phi}}{\partial \varphi_i \partial \varphi_j} \right)_k, \\ \ell_{ij,k}^h &= - \left\{ \frac{\ell_{j,k+1}^{\bar{\phi}} - \ell_{j,k-1}^{\bar{\phi}} - \ell_{j,k+1}^{\bar{\phi}} + \ell_{j,k-1}^{\bar{\phi}}}{4(\Delta\varphi_i)^2} \right\}, \end{aligned} \quad (2-43)$$

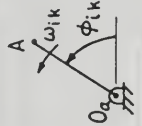
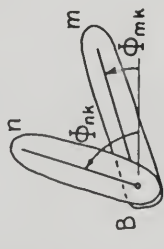
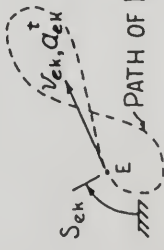
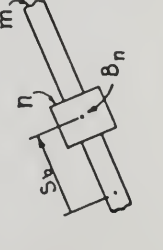
where the input type (i.e., angular or linear) is considered understood and its designation is dropped for simplification of notation as follows:

$$\begin{aligned} \mathcal{L}_{i,k}^g &\equiv \left(\bar{\varphi}_l^g \varphi_i \right)_k, \\ \mathcal{L}_{ij,k}^h &\equiv \left(\bar{\varphi}_l^h \varphi_i \varphi_j \right)_k. \end{aligned} \quad (2-44)$$

Table 2-1 is a compact collection of the influence coefficients derived thus far, expressed in discrete form. Each column is headed by a characteristic set of input conditions and each row is designated by a particular system model. Rows and columns intersect in blocks containing the influence coefficients for which these properties hold. Blocks 1.1 - 1.3 and 3.1 - 3.3 have been derived and are expressed in discrete form.

The coefficients in blocks 2.1 - 2.3 pertain to the relative angular motion between any two links m,n in the system (where m,n are not to be confused with m,n denoting the number of inputs and outputs as in Equation (2-41)). The links are shown with coincident points at B for purposes of simplicity of notation. The influence coefficients in blocks 4.1 - 4.3 apply to the relative motion between any two points B_m, B_n in different links m,n in the system. The system model has been represented as a sliding pair connecting these two points. It may be helpful to consider this pair as massless when it is not required for constraint in the system. An example of when it would be required for constraint is that shown in Figure 2-6.

The influence coefficients in Table 2-1 are a precise definition of the system geometry up to the second order. As such they

INPUT SYSTEM MODEL	VELOCITY	ACCELERATION	
 w_{ik}, α_{ik} $i = 1, \dots, n$ $w_{jk} = 0$ $j = 1, \dots, l-1, l+1, \dots, n$	1.1 $w_{jk} = \sum_{i=1}^n g_{jk} w_{ik}$ $g_{jk} = \frac{\Phi_{ik, k-1} - \Phi_{ik-1, k-1}}{2 \Delta \Phi_i}$	1.2 $\alpha_{jk} = \sum_{i=1}^n g_{jk} \alpha_{ik}$	1.3 $\alpha_{jk} = \sum_{i=1}^n \sum_{j=1}^n h_{ij, k} w_{ik} w_{jk}$ $h_{ij, k} = - \left\{ \frac{\Phi_{ij, k-1}^2 - \Phi_{ij, k-1}^2 - \Phi_{ij, k-1}^2 + \Phi_{ij, k-1}^2}{4 \Delta \Phi_i \Delta \Phi_j} \right\}$
 w_{ik}, α_{ik} $i = 1, \dots, n$ $w_{jk} = 0$ $j = 1, \dots, l-1, l+1, \dots, n$	2.1 $w_{bk} = \sum_{i=1}^n b_{ik} w_{ik}$ $b_{ik} = n g_{ik} - m g_{ik}$	2.2 $\alpha_{bk} = \sum_{i=1}^n b_{ik} \alpha_{ik}$	2.3 $\alpha_{bk} = \sum_{i=1}^n \sum_{j=1}^n b_{ij, k} w_{ik} w_{jk}$ $b_{ij, k} = h_{ij, k} - m_{ij, k}$
 w_{ik}, α_{ik} $i = 1, \dots, n$ $w_{jk} = 0$ $j = 1, \dots, l-1, l+1, \dots, n$	3.1 $v_{ek} = \sum_{i=1}^n \bar{g}_{ik} w_{ik}$ $\bar{g}_{ik} = \frac{e S_{ik, k-1} - e S_{ik-1, k-1}}{2 \Delta \Phi_i}$	3.2 $\alpha_{ek}^t = \sum_{i=1}^n \bar{g}_{ik} \alpha_{ik}$	3.3 $\alpha_{ek}^t = \sum_{i=1}^n \sum_{j=1}^n h_{ij, k} w_{ik} w_{jk}$ $h_{ij, k} = - \left\{ \frac{S_{ij, k-1}^2 - S_{ij, k-1}^2 - S_{ij, k-1}^2 + S_{ij, k-1}^2}{4 \Delta \Phi_i \Delta \Phi_j} \right\}$
 w_{ik}, α_{ik} $i = 1, \dots, n$ $w_{jk} = 0$ $j = 1, \dots, l-1, l+1, \dots, n$	4.1 $v_{bk} = \sum_{i=1}^n b_{ik} w_{ik}$ $b_{ik} = \frac{b S_{ik, k-1} - b S_{ik-1, k-1}}{2 \Delta \Phi_i}$	4.2 $\alpha_{bk}^t = \sum_{i=1}^n b_{ik} \alpha_{ik}$	4.3 $\alpha_{bk}^t = \sum_{i=1}^n \sum_{j=1}^n b_{ij, k} w_{ik} w_{jk}$ $b_{ij, k} = - \left\{ \frac{S_{ij, k-1}^2 - S_{ij, k-1}^2 - S_{ij, k-1}^2 + S_{ij, k-1}^2}{4 \Delta \Phi_i \Delta \Phi_j} \right\}$

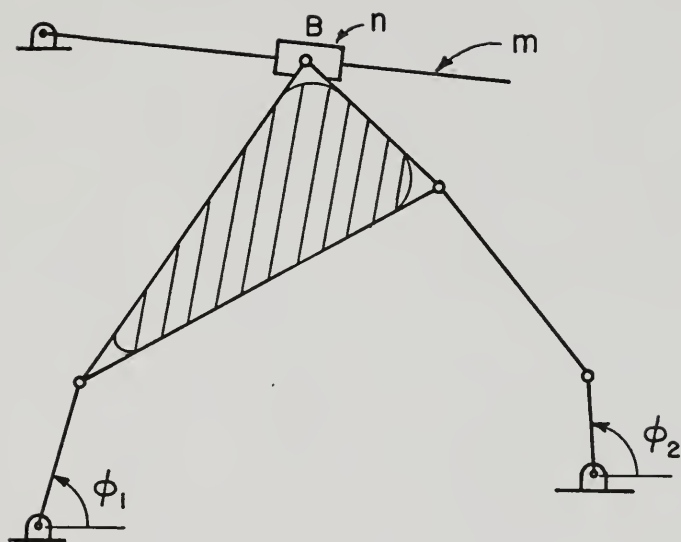


Figure 2-6. Sliding Pair Constraint

represent a powerful definition of the meaning of multiple input linkage systems.

These influence coefficients enable one to determine the dynamic state of every link in a general linkage system, given the dynamic state of the system inputs. They will be used in the following chapter to reduce complex, multiple input linkage systems acted upon by external forces and torques, internal springs, and viscous dampers, to equivalent mass systems acted upon by equivalent external torques, variable rate springs and variable coefficient viscous dampers.

The remaining coefficients will be treated on a discrete time basis, since some of them will depend on their past history. This means that expressions such as that for S_k imply that

$$\dot{S} = f(t) . \quad (2-45)$$

CHAPTER III

EQUIVALENT SYSTEM FORMULATION IN TERMS OF KINEMATIC INFLUENCE COEFFICIENTS

The coefficients developed in Chapter II will now be utilized to eliminate the holonomic constraints associated with complex linkage systems, reducing them to coupled, equivalent mass systems. These same coefficients will be used to reduce the generalized internal and external force generators acting on the system to equivalent generalized torques acting at the system inputs.

The Equivalent System Torques

The force related influence coefficients are derived by replacing the effect of external forces on the system (i.e., $\bar{T}_\ell$, $\bar{F}_e$) and internal forces generated by system elements, such as springs and viscous dampers, with equivalent torques acting on the n system inputs.

External Forces and Torques

If a linkage system (see Figure 3-1(a)) is given a virtual displacement by each separate input from some given system position k , then the virtual work done by the equivalent torque at the i^{th} input must be equal to the virtual work done by the system external torque, or

$$\left(T_{\ell/i} \Delta \varphi_i \right)_k = \left(\bar{T}_\ell \Delta \bar{\varphi}_\ell \right)_k, \quad (3-1)$$

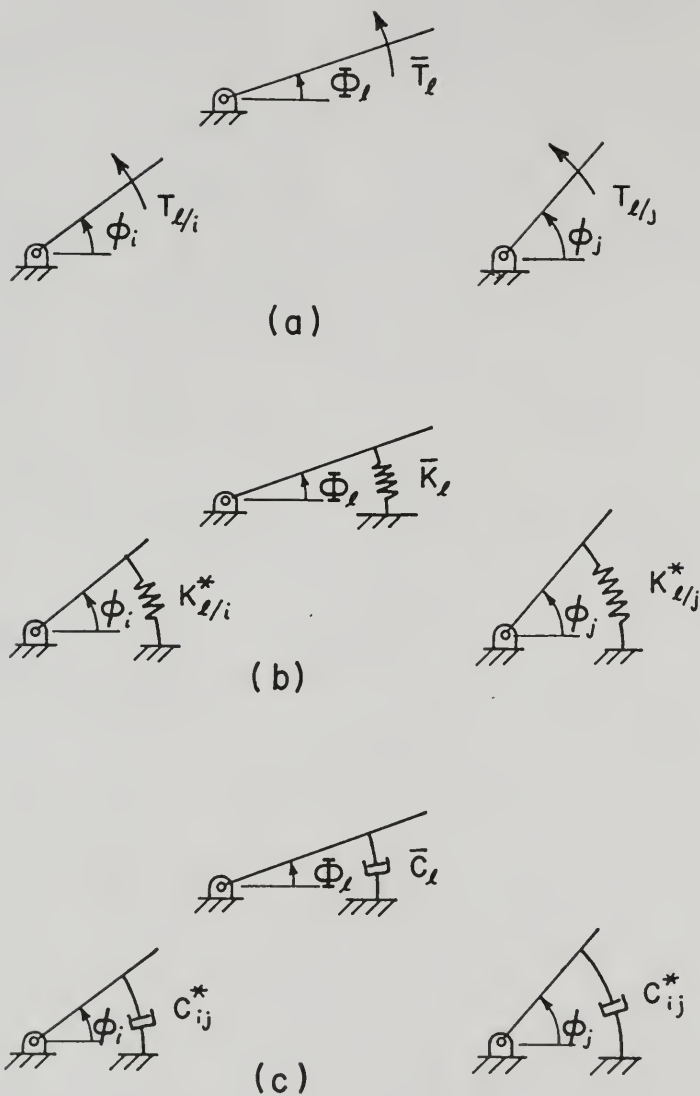


Figure 3-1. Equivalent System Elements

solving for $T_{l/ik}$ and taking the limit as $\Delta\varphi_{ik} \rightarrow 0$ gives

$$T_{l/ik} = \lim_{\Delta\varphi_i \rightarrow 0} \left(\frac{\Delta\bar{\varphi}_l}{\Delta\varphi_i} \right)_k \bar{T}_{lk} = (l^G_{ik}) \bar{T}_{lk} . \quad (3-2)$$

Similarly, for an external system force $\bar{F}_{ek}$,

$$T_{l/ik} = (e^G_{ik}) F_{ek} . \quad (3-3)$$

The total equivalent torque seen at each input link i , resulting from all system torques $\bar{T}_{lk}$, is given by

$$\begin{bmatrix} T_{e_1}^* \\ \vdots \\ T_{e_n}^* \end{bmatrix}_k = \begin{bmatrix} 1^G_1 & 2^G_1 & \cdots & N^G_1 \\ 1^G_2 & 2^G_2 & \cdots & N^G_2 \\ \vdots & \vdots & & \vdots \\ 1^G_n & 2^G_n & \cdots & N^G_n \end{bmatrix}_k \begin{bmatrix} \bar{T}_1 \\ \vdots \\ \bar{T}_N \end{bmatrix}_k = \begin{bmatrix} 1^G_i \\ \vdots \\ N^G_i \end{bmatrix}_k \begin{bmatrix} \bar{T}_1 \\ \vdots \\ \bar{T}_N \end{bmatrix}_k , \quad (3-4)$$

$$i = 1, 2, \dots, n,$$

$$N = \text{number of system links.}$$

Equations (3-2) and (3-3) show that l^G_{ik} and e^G_{ik} are the influence of a unit torque or force applied to system link l or point E on the i^{th} system input link. The l^G_{ik} is the mechanical advantage of link l with respect to the i^{th} input link and vice versa.

Internal Springs

Perhaps the most difficult system parameter to develop in terms of an equivalent coefficient is the equivalent spring constant $K_{l/ik}^*$. Suppose a torque $\bar{T}_{lk}$ is generated by a linear spring $\bar{K}_l$ between link l

and the ground link. The question arises: What is the equivalent spring constant $K_{\ell/ik}^*$ of the system spring $\bar{K}_\ell$? By giving the system a virtual displacement, the potential energy change in the system spring can be equated with the potential energy in the equivalent spring associated with the i^{th} input link (see Figure 3-2(b)). This yields the following relation

$$\frac{1}{2} K_{\ell/ik}^* (\Delta \varphi_{ik})^2 = \frac{1}{2} \bar{K}_\ell (\Delta \bar{\varphi}_{ik})^2. \quad (3-5)$$

Solving for $K_{\ell/ik}^*$ and taking the limit as $\Delta \varphi_{ik} \rightarrow 0$ yields

$$K_{\ell/ik}^* = \bar{K}_\ell \lim_{\Delta \varphi_{ik} \rightarrow 0} \left(\frac{\Delta \bar{\varphi}_\ell}{\Delta \varphi_{ik}} \right)^2 = \ell_{ik}^2 \bar{K}_\ell. \quad (3-6)$$

It is noted that $\bar{\varphi}_\ell$ cannot continuously increase with increasing φ_i , otherwise the spring would be destroyed. Rather, $\bar{\varphi}_\ell$ is cyclic or quasi-cyclic with increasing φ_i . Therefore, for part of the cycle $\Delta \bar{\varphi}_\ell$ (i.e., ℓ_{ik}^2) must be negative. This implies the need for the signum function $\sigma_{\ell_{ik}^2} = \text{sgn}(\ell_{ik}^2)$ which changes sign whenever the ℓ_{ik}^2 's go through zero. The torque $T_{\ell/ik}$ is then determined from its past history in the form

$$T_{\ell/ik} = T_{\ell/i,k-1} + K_{\ell/ik}^* \Delta \varphi_{ik}, \quad (3-7)$$

where

$$K_{\ell/ik}^* = \sigma_{\ell_{ik}^2} \bar{K}_\ell \left(\frac{\ell_{ik-1}^2 + \ell_{ik}^2}{2} \right). \quad (3-8)$$

The equivalent torques at each system input resulting from all system springs are given in matrix form as

$$\begin{bmatrix} T_{s/1}^* \\ T_{s/2}^* \\ \vdots \\ T_{s/n}^* \end{bmatrix}_k = \begin{bmatrix} T_{s/1}^* \\ T_{s/2}^* \\ \vdots \\ T_{s/n}^* \end{bmatrix}_{k-1} + \sum_{\ell=1}^N \begin{bmatrix} K_{\ell/1}^* \Delta \phi_1 \\ K_{\ell/2}^* \Delta \phi_2 \\ \vdots \\ K_{\ell/n}^* \Delta \phi_n \end{bmatrix}_k \quad (3-9)$$

An alternative approach to determine the equivalent torque $T_{\ell/ik}$ generated by a system spring between link ℓ and ground is to treat the torque acting on link ℓ due to spring $\bar{K}_\ell$ as an external torque as in Equation (3-2). The external torque is given by

$$\bar{T}_{\ell k} = \bar{K}_\ell \left(\bar{\phi}_{\ell f} - \bar{\phi}_{\ell k} \right) = \bar{K}_\ell \Delta \bar{\phi}_{\ell k}^* \quad (3-10)$$

where

$\bar{\phi}_{\ell f}$ = free length of spring measured from the reference axis,

$\Delta \bar{\phi}_{\ell k}^*$ = deformation of system spring $\bar{K}_\ell$ from free length.

Substituting Equation (3-10) into (3-2) gives

$$T_{\ell/ik} = \bar{K}_\ell \left(\ell_{ik}^g \right) \Delta \bar{\phi}_{ik}^* \quad (3-11)$$

The equivalent torques at each system input resulting from all system springs $\bar{K}_\ell$ are now given in matrix form as

$$\begin{bmatrix} T_{s/1}^* \\ \vdots \\ T_{s/n}^* \end{bmatrix}_k = \begin{bmatrix} 1 G_i \\ \vdots \\ N G_i \end{bmatrix}_k \begin{bmatrix} \bar{K}_1 \Delta \bar{\phi}_1^* \\ \vdots \\ \bar{K}_N \Delta \bar{\phi}_N^* \end{bmatrix}_k, \quad i = 1, 2, \dots, n, \quad (3-12)$$

N = number of system links.

Internal Viscous Dampers

Let the torque $\bar{T}_{\ell k}$ be generated by a system dashpot $\bar{C}_{\ell}$ between link ℓ and ground (see Figure 3-1(c)). The torque generated by the dashpot is proportional to the angular velocity of the link attached to the dashpot, or

$$\bar{T}_{\ell k} = \bar{C}_{\ell} \sum_{j=1}^n \left(\ell_{\ell j}^g \omega_j \right)_k . \quad (3-13)$$

Using Equation (3-2), $\bar{T}_{\ell k}$ is transferred to the i^{th} system input as

$$T_{\ell/i k} = \left(\ell_{\ell i k}^g \right) \bar{C}_{\ell} \sum_{j=1}^n \left(\ell_{\ell j}^g \omega_j \right)_k . \quad (3-14)$$

The total torque acting on the i^{th} input link resulting from all system dashpots is

$$T_{d/i k}^* = \sum_{j=1}^n \left(c_{ij}^* \omega_j \right)_k , \quad (3-15)$$

where

$$c_{ij,k}^* = \sum_{\ell=1}^N \left(\bar{C}_{\ell} \ell_{\ell i}^g \ell_{\ell j}^g \right)_k . \quad (3-16)$$

The equivalent torques at all system inputs resulting from all system dashpots are given in matrix form as

$$\begin{bmatrix} T_{d/1}^* \\ T_{d/2}^* \\ \vdots \\ T_{d/n}^* \end{bmatrix}_k = \begin{bmatrix} c_{11}^* & c_{12}^* & \cdots & c_{1n}^* \\ c_{21}^* & c_{22}^* & \cdots & c_{2n}^* \\ \vdots & \vdots & & \vdots \\ c_{n1}^* & c_{n2}^* & \cdots & c_{nn}^* \end{bmatrix}_k \left| \omega \right|_k . \quad (3-17)$$

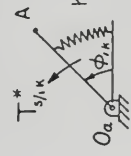
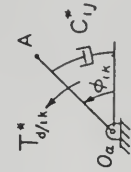
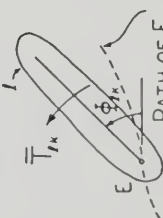
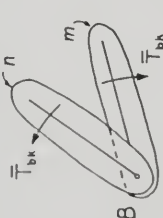
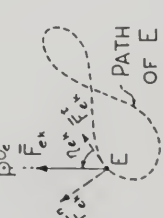
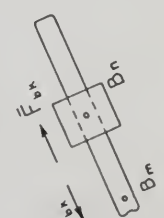
Table 3-1 is a concise collection of the preceding equivalent torque concepts. Row 2 corresponds directly with row 1. Those in row 2 are more general, since the generated torques occur between two moving links m and n in the system. The relative influence coefficients b_{ik}^r correspond to those in Table 2-1, i.e.,

$$b_{ik}^r = n_{ik}^g - m_{ik}^g \quad (3-18)$$

The equivalent torques in row 3 refer to a force $\bar{F}_{ek}$ applied to a particular point or points in the system. Since the direction angle η_{ek} of this force relative to the path tangent is known as a function of the input angles φ_i , then the appropriate equivalent torque can be determined. The equivalent torque, $T_{e/ik}$, is obtained by first taking its component $\bar{F}_{ek}^t$ along the path tangent and transferring this force to the i^{th} input link by the influence coefficient $\bar{g}_{ik}$. Here, the product $\bar{g}_{ik} \cos(\eta_{ek})$ is the effective influence of a unit force applied in the direction $\bar{F}_{ek}$ at point E on the i^{th} input link. As such, it corresponds directly with g_{ik} in row 1. Blocks 3.2 and 3.3 follow directly as a result of this correspondence. Here, the system spring $\bar{K}_e$ and dashpot $\bar{C}_e$ are anchored at some fixed point O_e .

Finally, the equivalent torques in row 4 apply to forces in extensible two-force members attached to links m and n at points B_m and B_n . The relative influence coefficient b_{ik}^r is defined in Table 2-1, row 4. This implies the treatment of very general force generators in the system.

TABLE 3-1: EQUIVALENT SYSTEM FORCES AND TORQUES

INPUT SYSTEM MODEL	SYSTEM TORQUE		SYSTEM SPRING		SYSTEM DASHPOT	
						
	1.1		1.2		1.3	
	$T_{e/1,k}^* = \sum_l T_{l/1,k}$ $T_{l/1,k} = (g_l \phi_{l,k}) \bar{T}_{l,k}$		$T_{s/1,k}^* = \sum_{i=1}^N \{ T_{i/1,k} + K_{i/1,k}^* \Delta \phi_{i,k} \}$ $K_{i/1,k}^* = \sigma_{i/1,k} \bar{K}_i \left(\frac{\Delta g_{i,k}^2 + \Delta g_{i,k}^2}{2} \right)$		$T_{d/1,k}^* = \sum_{j=1}^n (C_{j/1,k}^* \dot{w}_j)_k$ $C_{j/1,k}^* = \sum_{i=1}^N (\bar{C}_i g_i g_j)_k$	
	2.1		2.2		2.3	
	$T_{e/1,k}^* = \sum_b T_{b/1,k}$ $T_{b/1,k} = (b_l \phi_{l,k}) \bar{T}_{b,k}$		$T_{s/1,k}^* = \sum_b \{ T_{b/1,k-1} + K_{b/1,k}^* \Delta \phi_{b,k} \}$ $K_{b/1,k}^* = \sigma_{b/1,k} \bar{K}_b \left(\frac{b_l \phi_{l,k-1}^2 + b_l \phi_{l,k}^2}{2} \right)$		$T_{d/1,k}^* = \sum_j (C_{j/1,k}^* \dot{w}_j)_k$ $C_{j/1,k}^* = \sum_b (\bar{C}_b \phi_{b,l} \phi_{b,j})_k$	
	3.1		3.2		3.3	
	$T_{e/1,k}^* = \sum_e T_{e/1,k}$ $T_{e/1,k} = (g_e \phi_{e,k}) \cos \eta_{e,k} \bar{F}_{e,k}$		$T_{s/1,k}^* = \sum_e \{ T_{e/1,k-1} + K_{e/1,k}^* \Delta \phi_{e,k} \}$ $K_{e/1,k}^* = \sigma_{e/1,k} \bar{K}_e \left(\frac{e_l \phi_{l,k-1}^2 \cos^2 \eta_{e,k} + e_l \phi_{l,k}^2 \cos^2 \eta_{e,k}}{2} \right)$		$T_{d/1,k}^* = \sum_j (C_{j/1,k}^* \dot{w}_j)_k$ $C_{j/1,k}^* = \sum_e (\bar{C}_e g_e g_j \cos^2 \eta_{e,k})_k$	
	4.1		4.2		4.3	
	$T_{e/1,k}^* = \sum_b T_{b/1,k}$ $T_{b/1,k} = (b_l \phi_{l,k}) \bar{F}_{b,k}$		$T_{s/1,k}^* = \sum_b \{ T_{b/1,k-1} + K_{b/1,k}^* \Delta \phi_{b,k} \}$ $K_{b/1,k}^* = \sigma_{b/1,k} \bar{K}_b \left(\frac{b_l \phi_{l,k-1}^2 + b_l \phi_{l,k}^2}{2} \right)$		$T_{d/1,k}^* = \sum_j (C_{j/1,k}^* \dot{w}_j)_k$ $C_{j/1,k}^* = \sum_b (\bar{C}_b \phi_{b,l} \phi_{b,j})_k$	

Total Equivalent Torque

The total equivalent torque acting upon the i^{th} input link is given as

$$T_{ik}^* = T_{l/ik}^* + T_{e/ik}^* + T_{s/ik}^* + T_{d/ik}^* \quad (3-19)$$

If one or more of the input elements is actually a sliding pair, then s_{ik} represents that input's reference position parameter and F_{ik}^* would be the equivalent force acting on that system input.

Equivalent System Inertias

The objective of this section is to transfer the effective mass of link l to each of the n system inputs in order to obtain equivalent mass systems for a linkage system such as that shown in Figure 3-2.

Consider the kinetic energy of a complex, multiple input system.

$$(KE)_k = \frac{1}{2} \sum_{l=1}^N \left\{ \bar{I}_l \omega_{lk}^2 + \bar{M}_l v_{ek}^2 \right\}, \quad (3-20)$$

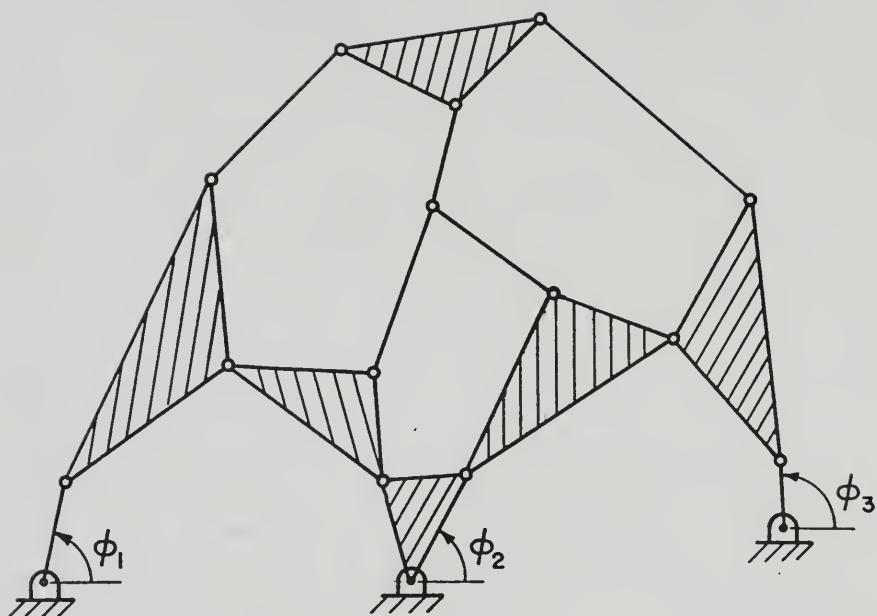
where

$\bar{I}_l$ = effective moment of inertia of link l about its center of gravity E ,

$\bar{M}_l$ = effective mass of link l located at its center of gravity E .

Writing Equation (3-20) in terms of the system input velocities yields

$$(KE)_k = \frac{1}{2} \sum_{l=1}^N \left\{ \bar{I}_l \left[\sum_{i=1}^n ({}_l g_{ik}) \omega_{ik} \right]^2 + \bar{M}_l \left[\sum_{i=1}^n ({}_e \bar{g}_{ik}) \omega_{ik} \right]^2 \right\}. \quad (3-21)$$



$$n = 3$$
$$N = 13$$

Figure 3-2. Complex Multiple Input Linkage System

Rewriting Equation (3-21) in a more compact form gives

$$(KE)_k = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{\ell=1}^N \left\{ \left[\bar{I}_{\ell}(\ell^g_i)(\ell^g_j) + \bar{M}_{\ell}(e^g_i)(e^g_j) \right] \omega_i \omega_j \right\}_k, \quad (3-22)$$

where it is noted that

$$(\ell^g_i)(\ell^g_j) = \begin{cases} 1 & \ell = i, j; \quad i = j \\ 0 & \ell = i, j; \quad i \neq j, \end{cases} \quad (3-23)$$

a direct consequence of the meaning of independent system inputs.

The kinetic energy can be written in matrix form as

$$(KE)_k = \frac{1}{2} |\omega|'_k \begin{bmatrix} I^*_{ij} \\ (n \times n) \end{bmatrix}_k |\omega|_k, \quad (3-24)$$

where

$$I^*_{ij,k} = \sum_{\ell=1}^N \left[\bar{I}_{\ell}(\ell^g_i)(\ell^g_j) + \bar{M}_{\ell}(e^g_i)(e^g_j) \right]_k. \quad (3-25)$$

Matrix Equation (3-24) is recognized as being real quadratic form

since the $[I^*_{ij}]$ matrix is real symmetric. Thus it is seen that any

complex multiple input linkage system can be transformed into a system of coupled equivalent masses with n degrees of freedom whose equivalent inertias are given by Equation (3-25).

Two Degrees of Freedom Example

For the purpose of illustration, consider the differential gear (Figure 3-3), where the translating links are constrained to move in the horizontal direction and rotation of link ℓ_4 is positive counterclockwise.

The linear velocity influence coefficients of ℓ_3 are

$${}^3\bar{g}_1 = \frac{\partial x_3}{\partial x_1} = \frac{d_2}{d_1 + d_2} \quad , \quad (3-26)$$

$${}^3\bar{g}_2 = \frac{\partial x_3}{\partial x_2} = \frac{d_1}{d_1 + d_2} \quad ,$$

and

$$v_3 = {}^3\bar{g}_1 v_1 + {}^3\bar{g}_2 v_2 \quad . \quad (3-27)$$

The angular velocity influence coefficients of ℓ_4 are given

by (see Figures 3-4 and 3-5)

$${}^4\bar{g}_1 = \frac{\partial \phi_4}{\partial x_1} = - \frac{1}{d_1 + d_2} \quad , \quad (3-28)$$

$${}^4\bar{g}_2 = \frac{\partial \phi_4}{\partial x_2} = + \frac{1}{d_1 + d_2} \quad ,$$

and

$$\omega_4 = {}^4\bar{g}_1 v_1 + {}^4\bar{g}_2 v_2 \quad . \quad (3-29)$$

The kinetic energy of the two input differential gear is

$$KE = \frac{1}{2} \{ \bar{M}_1 v_1^2 + \bar{M}_2 v_2^2 + (\bar{M}_3 + \bar{M}_4) v_3^2 + \bar{I}_4 \omega_4^2 \} \quad . \quad (3-30)$$

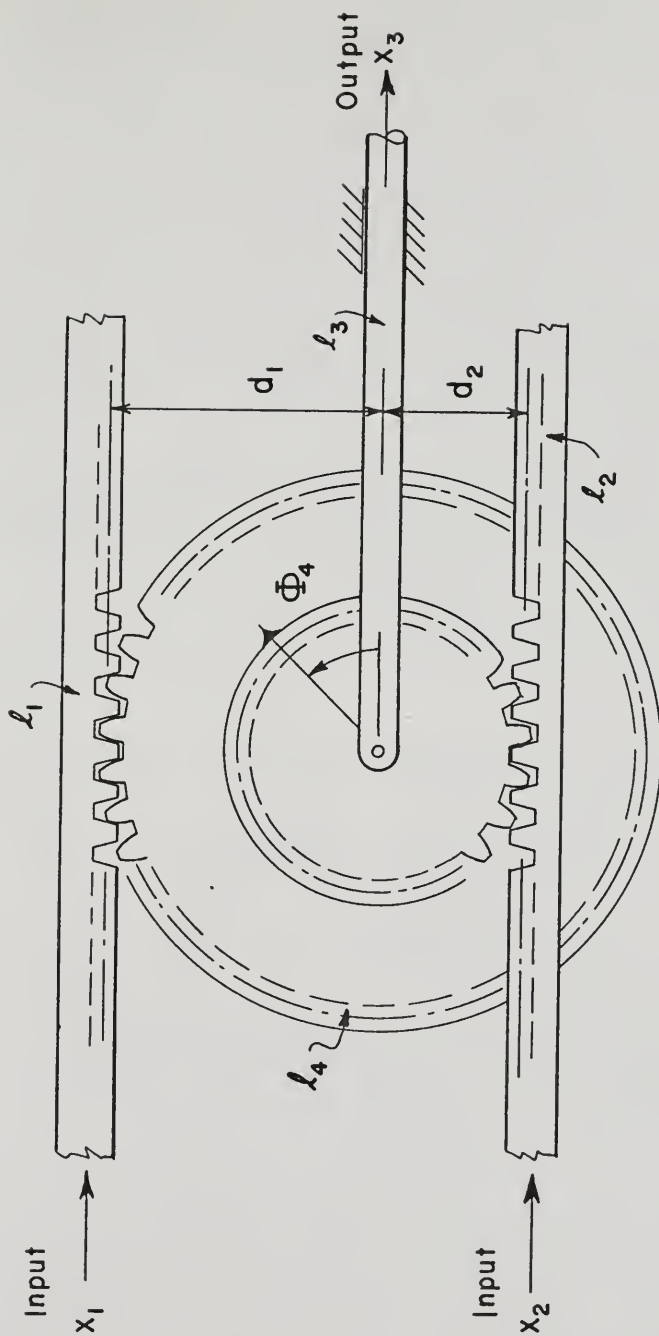
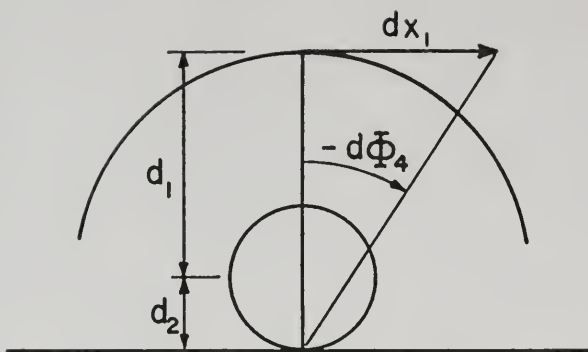
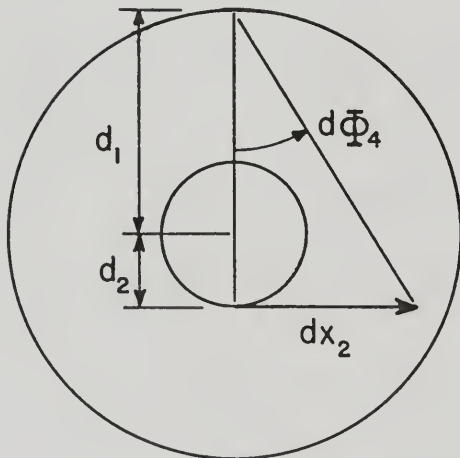


Figure 3-3. Differential Gear System



$$dx_1 = (d_1 + d_2)(-d\Phi_4)$$

Figure 3-4. Angular Relationship to Input #1



$$dx_2 = (d_1 + d_2) d\Phi_4$$

Figure 3-5. Angular Relationship to Input #2

Substituting Equations (3-27) and (3-29) into (3-30) gives

$$\begin{aligned} KE = \frac{1}{2} \{ \bar{M}_1 v_1^2 + \bar{M}_2 v_2^2 + (\bar{M}_3 + \bar{M}_4) (\bar{g}_1 v_1 + \bar{g}_2 v_2)^2 \\ + \bar{I}_4 ({}_4g_1 v_1 + {}_4g_2 v_2)^2 \} , \end{aligned} \quad (3-31)$$

where

$$v_3 = v_4 .$$

Collecting like terms and rewriting Equation (3-31) gives

$$\begin{aligned} KE = \frac{1}{2} \{ [\bar{M}_1 + \bar{I}_4 ({}_4g_1)^2 + (\bar{M}_3 + \bar{M}_4) (\bar{g}_1)^2] v_1^2 \\ + [\bar{M}_2 + \bar{I}_4 ({}_4g_2)^2 + (\bar{M}_3 + \bar{M}_4) (\bar{g}_2)^2] v_2^2 \\ + 2[\bar{I}_4 ({}_4g_1) ({}_4g_2) + (\bar{M}_3 + \bar{M}_4) (\bar{g}_1) (\bar{g}_2)] v_1 v_2 \} , \end{aligned} \quad (3-32)$$

or

$$KE = \frac{1}{2} \{ I_{11}^* v_1^2 + I_{22}^* v_2^2 + 2I_{12}^* v_1 v_2 \} , \quad (3-33)$$

where

$$\begin{aligned} I_{11}^* &= \bar{M}_1 + \bar{I}_4 ({}_4g_1)^2 + (\bar{M}_3 + \bar{M}_4) (\bar{g}_1)^2 , \\ I_{12}^* &= \bar{I}_4 ({}_4g_1) ({}_4g_2) + (\bar{M}_3 + \bar{M}_4) (\bar{g}_1) (\bar{g}_2) , \\ I_{22}^* &= \bar{M}_2 + \bar{I}_4 ({}_4g_2)^2 + (\bar{M}_3 + \bar{M}_4) (\bar{g}_2)^2 . \end{aligned} \quad (3-34)$$

For the system in Figure 3-3, Equation (3-24) becomes

$$KE = \frac{1}{2} [v_1 \quad v_2] \begin{bmatrix} I_{11}^* & I_{12}^* \\ I_{21}^* & I_{22}^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} , \quad (3-35)$$

which is recognized as a quadratic form.

In the case of a differential gear the equivalent inertias are constant. However, for a general linkage system (Figure 3-2), the I^* 's are not constant but a function of the system input parameters. Note that whenever $v_2 = 0$, then

$$KE = \frac{1}{2} I_{11}^* v_1^2, \quad (3-36)$$

and whenever $v_1 = 0$, then

$$KE = \frac{1}{2} I_{22}^* v_2^2, \quad (3-37)$$

which are simply the expressions for the kinetic energy of a single input equivalent system.

Inertia coupling terms, I_{ij}^* $i \neq j$, appear for systems with two or more inputs, indicating a relationship between the I_{ij}^* 's through the velocity product terms $\omega_i \omega_j$. Quinn's energy method [29] becomes invalid for these types of systems, since the distribution of kinetic energy between the links of the system is no longer invariant with respect to the input velocities. The equivalent mass system described above remains valid provided certain modifications are made with respect to the solution technique.

Equivalent System Inertia Power

The final equivalent system parameter to be considered is the equivalent system inertia power. The coefficients derived in this section are necessary when deriving the differential equations of motion of complex linkage systems. If the I_{ij}^* 's were constant, as in the preceding example, these terms would be zero. Although they appear

unwieldy in size and computation for systems with more than two inputs, they readily lend themselves to computer computation.

The power necessary to drive a linkage system against its inertias is given as

$$P^i = \frac{d}{dt} (KE) , \quad (3-38)$$

where

$$KE = \frac{1}{2} |\omega|' \left[I_{ij}^* \right] |\omega| , \quad i, j = 1, 2, \dots, n . \quad (3-24)$$

Differentiating Equation (3-24) with respect to time gives

$$P^i = \frac{1}{2} |\omega|' \frac{d}{dt} \left[I_{ij}^* \right] |\omega| + |\alpha|' \left[I_{ij}^* \right] |\omega| . \quad (3-39)$$

The second product on the right side of Equation (3-39) is self-explanatory. The time derivative of the equivalent inertia matrix, however, is not as trivial. Consider a typical element of $[I_{ij}^*]$ given by

$$I_{ij}^* = \sum_{\ell=1}^N \left[\bar{I}_{\ell} (\ell^g_i) (\ell^g_j) + \bar{M}_{\ell} (e^g_i) (e^g_j) \right] . \quad (3-40)$$

Differentiating Equation (3-40) with respect to time gives

$$\begin{aligned} \left(\frac{dI_{ij}^*}{dt} \right)_k &= \sum_{r=1}^n \sum_{\ell=1}^N \left\{ \bar{I}_{\ell} \left[(\ell^g_i) (\ell^h_{jr}) + (\ell^g_j) (\ell^h_{ir}) \right] \right. \\ &\quad \left. + \bar{M}_{\ell} \left[(e^g_i) (e^h_{jr}) + (e^g_j) (e^h_{ir}) \right] \right\} \omega_{rk} , \end{aligned} \quad (3-41)$$

where

$$(\ell^g_i) (\ell^h_{jr}) = \frac{\partial \bar{I}_{\ell}}{\partial \varphi_i} \frac{\partial^2 \bar{I}_{\ell}}{\partial \varphi_j \partial \varphi_r} . \quad (3-42)$$

Equation (3-41) can be written as the product of a row and column vector as

$$\left(\frac{dI_{ij}^*}{dt}\right)_k = \begin{bmatrix} i_j P_1^* & \cdots & i_j P_n^* \end{bmatrix} \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}, \quad (3-43)$$

where

$$i_j P_r^* = \sum_{l=1}^N \left\{ \bar{I}_l \left[(g_i)(h_{jr}) + (g_j)(h_{ir}) \right] + \bar{M}_l \left[(e_i)(h_{jr}) + (e_j)(h_{ir}) \right] \right\}_k. \quad (3-44)$$

The subscript notation is defined by

$$i_j P_r^* \triangleq \frac{\partial I_{ij}^*}{\partial \varphi_r}. \quad (3-45)$$

Substituting Equation (3-43) into Equation (3-39) gives

$$\begin{array}{ccc} (1 \times n) & (n \times n) & (n \times n) \\ P_k^i = \frac{1}{2} |\omega|'_k \begin{bmatrix} 11^{P_r^*} & 12^{P_r^*} & \cdots & 1n^{P_r^*} \\ 21^{P_r^*} & 22^{P_r^*} & \cdots & 2n^{P_r^*} \\ \vdots & \vdots & & \vdots \\ n1^{P_r^*} & n2^{P_r^*} & \cdots & nn^{P_r^*} \end{bmatrix} \begin{bmatrix} |\omega| & 0 & \cdots & 0 \\ 0 & |\omega| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & |\omega| \end{bmatrix} |\omega|_k \\ + |\alpha|'_k \begin{bmatrix} I_{ij}^* \end{bmatrix}_k |\omega|_k, & & (3-46) \end{array}$$

where $\begin{bmatrix} i_j P_r^* \end{bmatrix}$ is a compound $n \times n$ matrix whose submatrices are $1 \times n$ row vectors.

Two Input Example

Consider the system in Figure 3-6 whose kinetic energy is given by

$$(KE)_k = \frac{1}{2} [\omega_1 \ \omega_2]_k \begin{bmatrix} I_{11}^* & I_{12}^* \\ I_{21}^* & I_{22}^* \end{bmatrix}_k \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_k, \quad (3-47)$$

where

$$I_{ij,k}^* = \sum_{\ell=1}^8 \left[\bar{I}_{\ell} \ell^{\bar{g}}_i \ell^{\bar{g}}_j + \bar{M}_{\ell} \ell^{\bar{g}}_i \ell^{\bar{g}}_j \right]_k. \quad (3-48)$$

Differentiating Equation (3-47) with respect to time gives

$$\begin{aligned} \dot{P}_k^i = \frac{1}{2} [\omega_1 \ \omega_2]_k & \begin{bmatrix} 11^{P*} & 11^{P*} & 12^{P*} & 12^{P*} \\ \hline 21^{P*} & 21^{P*} & 22^{P*} & 22^{P*} \end{bmatrix}_k \begin{bmatrix} \omega_1 & 0 \\ \omega_2 & 0 \\ 0 & \omega_1 \\ 0 & \omega_2 \end{bmatrix}_k \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_k \\ & + [\alpha_1 \ \alpha_2]_k \begin{bmatrix} I_{11}^* & I_{12}^* \\ I_{21}^* & I_{22}^* \end{bmatrix}_k \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_k, \end{aligned} \quad (3-49)$$

where

$$12^{P*}_1 = \sum_{\ell=1}^8 \left[I_{\ell} (\ell^{\bar{g}}_1 \ell^{\bar{h}}_{11} + \ell^{\bar{g}}_2 \ell^{\bar{h}}_{21}) + \bar{M}_{\ell} (\ell^{\bar{g}}_1 \ell^{\bar{h}}_{11} + \ell^{\bar{g}}_2 \ell^{\bar{h}}_{21}) \right]_k. \quad (3-50)$$

Table 3-2 represents the concepts of equivalent system inertias and equivalent inertia power for complex, multiple input linkage systems. With this formulation the holonomic constraints are eliminated and the equations of motion derived, utilizing only those generalized coordinates associated with the independent inputs.

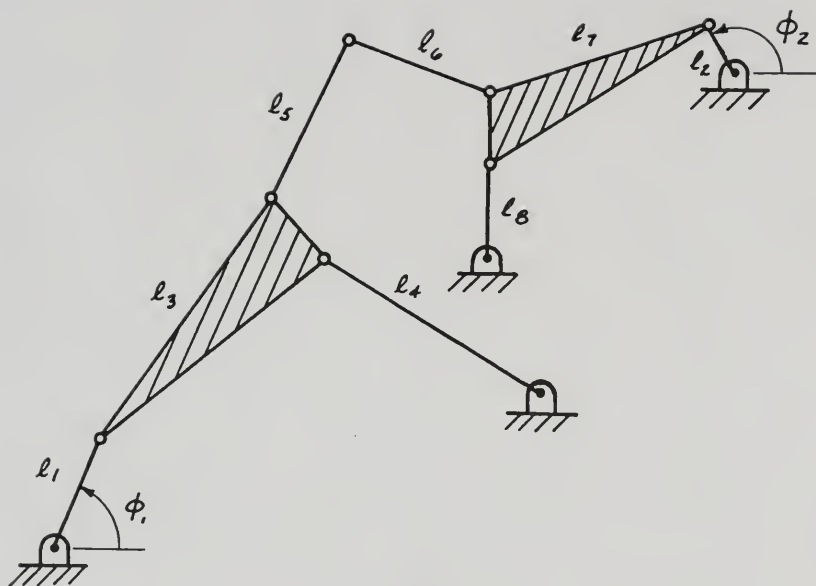
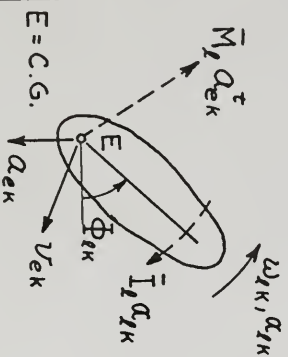
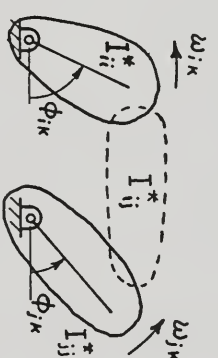
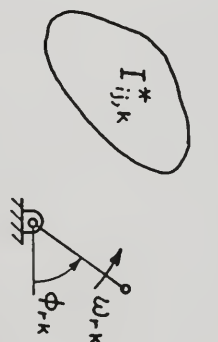


Figure 3-6. Complex Two Input Linkage System

TABLE 3-2: EQUIVALENT SYSTEM FORMULATION

INPUT CONDITION SYSTEM MODEL		
 $\bar{M}_l a_{ek}^{\tau}$ $E = C.G.$ a_{ek} v_{ek} Φ_{ek} $\bar{I}_l a_{lk}$ ω_{lk}, a_{lk}	 $I_{ij,k}^* = \sum_{l=1}^N \{ \bar{I}_l (a_{gi}) (a_{gi}) + \bar{M}_l (e_{gi}) (e_{gi}) \}_k$ 1.1	 $I_{ij,k}^*$ ϕ_{rk} ω_{rk} $i_j P_r^* = \frac{\partial I_{ij}^*}{\partial \phi_r}$ 1.2 $i_j P_{r,k}^* = \sum_{l=1}^N \{ \bar{I}_l [(g_l \chi_{h_l}) + (g_l \chi_{h_{l-r}})] + \bar{M}_l [(e_{gi} \chi_{h_l}) + (e_{gi} \chi_{h_{l-r}})] \}_k$

CHAPTER IV

TIME RESPONSE OF EQUIVALENT MASS SYSTEMS

Complex, multiple input linkage systems present a unique problem with respect to determining their dynamic response. Methods for determining the time response of single input systems are well established, either in terms of Lagrangian mechanics [11-14] or equivalent mass systems. Each method has its strengths and weaknesses. The method chosen to solve any given problem should be based on need and solution form. It is possible to obtain phase plane solutions for single input systems by using techniques in [1] if the input velocity is never less than zero. With slight modifications in the predictor equation, negative velocities can also be treated. Here the independent variable is the input angle, φ_1 , rather than time, a result of the cyclic nature of most mechanisms. However, a large class of problems does not meet this requirement. Hence, a more general method must be employed.

As pointed out earlier, the energy distribution method is invalid for multiple input systems. Similarly, the kinetic energy and power concepts utilizing the equivalent mass and force system developed in [1] fail because they yield only one second-order differential equation in n independent variables. These failings are a result of the geometric constraints imposed on the system.

Two methods will be discussed in this chapter. The differential equations of motion for a two input system will be derived, using both methods. The complete set of first-order difference equations will be derived for only the second method for reasons which will be explained later. The methods used to derive the differential equations and to solve them are not new. The kinematic influence coefficients and coupled equivalent systems (inertias) described in Chapters II and III are new and unique, enabling established methods to be used in deriving and solving the differential equations of motion.

Lagrange's Method

A system of n second-order differential equations is derived from Lagrange's equations as

$$\frac{d}{dt} \left(\frac{\partial L^*}{\partial \dot{\varphi}_i} \right) - \frac{\partial L^*}{\partial \varphi_i} = T_i^*, \quad (i = 1, 2, \dots, n) \quad (4-1)$$

where

φ_i - generalized coordinates,

$\dot{\varphi}_i$ - generalized velocities,

L^* - Lagrangian of the coupled equivalent mass system,

T_i^* - equivalent nonpotential forces and torques.

The n second-order differential equations resulting from Equation (4-1) are nonlinear, coupled, nonhomogeneous, and contain variable coefficients. It is necessary to reduce this set of equations to $2n$ first-order differential equations in order to solve them on a digital computer. Although this reduction is entirely possible, it is often tedious, yielding unwieldy equations. The variable coefficients

are functions of geometry and are hence implicit functions of time. This characteristic requires the determination of the equivalent inertia power coefficients of the system, terms which are very difficult to obtain (see Chapter III) and only adds to the complexity of the problem.

Example

The five-bar mechanism in Figure 4-1 is a representative two input system. The generalized coordinates are φ_1 and φ_2 ; a helical spring and viscous damper are attached between links 3 and 4 at pin joint B, while external torques T_1 and T_2 are applied to links 1 and 2, respectively.

The Lagrangian for this system is given as

$$L^* = (KE - PE), \quad (4-2)$$

and the equivalent nonconservative torques are

$$\begin{aligned} T_1^* &= -\bar{C}_{34} b_{r1} [b_{r1} \dot{\varphi}_1 + b_{r2} \dot{\varphi}_2] + T_1, \\ T_2^* &= -\bar{C}_{34} b_{r2} [b_{r1} \dot{\varphi}_1 + b_{r2} \dot{\varphi}_2] + T_2, \end{aligned} \quad (4-3)$$

where

$$KE = \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 I_{ij}^* \dot{\varphi}_i \dot{\varphi}_j, \quad (4-4)$$

$$PE = \frac{1}{2} \bar{K}_{34} (\varphi_{34f} - \varphi_{34})^2. \quad (4-5)$$

Substituting Equations (4-3), (4-4), and (4-5) into Equation (4-1) and performing the designated differentiation with respect to the generalized coordinates and velocities yields

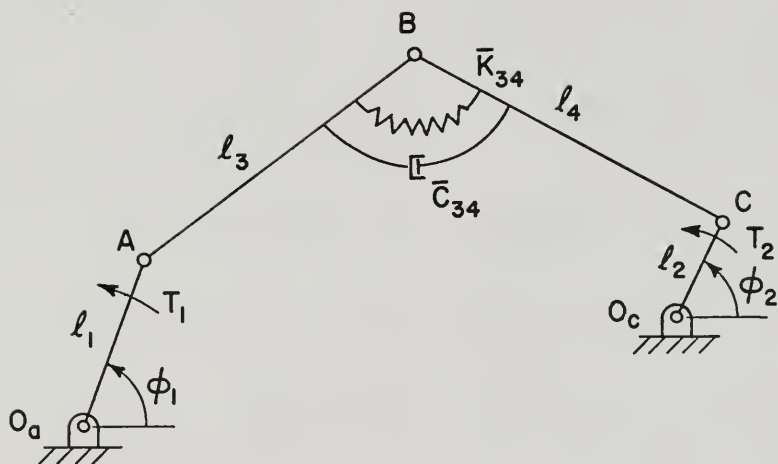


Figure 4-1. Two Input System

$$\begin{aligned} \frac{d}{dt} \left\{ I_{11}^* \dot{\varphi}_1 + I_{12}^* \dot{\varphi}_2 \right\} + \bar{K}_{34} (\varphi_{34f} - \varphi_{34}) ({}_b r_1) \\ = T_1 - \bar{C}_{34} [({}_b r_1^2) \dot{\varphi}_1 + ({}_b r_1) ({}_b r_2) \dot{\varphi}_2] , \end{aligned} \quad (4-6)$$

$$\begin{aligned} \frac{d}{dt} \left\{ I_{12}^* \dot{\varphi}_1 + I_{22}^* \dot{\varphi}_2 \right\} + \bar{K}_{34} (\varphi_{34f} - \varphi_{34}) ({}_b r_2) \\ = T_2 - \bar{C}_{34} [({}_b r_1) ({}_b r_2) \dot{\varphi}_1 + ({}_b r_2^2) \dot{\varphi}_2] . \end{aligned} \quad (4-7)$$

Completing the differentiation with respect to time gives

$$\begin{aligned} {}_{11}P_1^* \dot{\varphi}_1 + ({}_{11}P_2^* + {}_{12}P_1^*) \dot{\varphi}_1 \dot{\varphi}_2 + {}_{12}P_2^* \dot{\varphi}_2^2 + I_{11}^* \ddot{\varphi}_1 + I_{12}^* \ddot{\varphi}_2 - T_1 \\ + \bar{K}_{34} (\varphi_{34f} - \varphi_{34}) ({}_b r_1) = \bar{C}_{34} \left[({}_b r_1^2) \dot{\varphi}_1 + ({}_b r_1) ({}_b r_2) \dot{\varphi}_2 \right] , \end{aligned} \quad (4-8)$$

$$\begin{aligned} {}_{21}P_1^* \dot{\varphi}_1^2 + ({}_{21}P_2^* + {}_{22}P_1^*) \dot{\varphi}_1 \dot{\varphi}_2 + {}_{22}P_2^* \dot{\varphi}_2^2 + I_{12}^* \ddot{\varphi}_1 + I_{22}^* \ddot{\varphi}_2 - T_2 \\ + \bar{K}_{34} (\varphi_{34f} - \varphi_{34}) ({}_b r_2) = \bar{C}_{34} \left[({}_b r_1) ({}_b r_2) \dot{\varphi}_1 + ({}_b r_2^2) \dot{\varphi}_2 \right] . \end{aligned} \quad (4-9)$$

Equations (4-8) and (4-9) are nonlinear in the velocities and are unwieldy in their present form. In addition, they should be reduced to four first-order equations before they are numerically integrated. The ${}_{ij}P_r^*$ terms as defined by Equations (3-44) and (3-45) are not simple expressions, and require calculation at each integration step.

Hamilton's Principle

The method used in this section for deriving first-order difference equations by direct application of Hamilton's principle was developed by Vance and Sitchin [18]. The section on the derivation of the method from [18] is included in the Appendix for completeness.

The motion of a dynamical system is determined by solving the 3nN equations derived from

$$\frac{\partial S_1}{\partial \dot{\varphi}_{ik}} + \sum_{j=1}^N \lambda_{ij} \frac{\partial f_{ij}}{\partial \dot{\varphi}_{ik}} = - F_{ik}^* \Delta t, \quad (4-10)$$

$$\frac{\partial S_1}{\partial \dot{\varphi}_{ik}} + \sum_{j=1}^N \lambda_{ij} \frac{\partial f_{ij}}{\partial \dot{\varphi}_{ik}} = 0, \quad (4-11)$$

$$f_{ik} = \Delta \varphi_{ik} - \dot{\varphi}_{ik} \Delta t = 0, \quad (i = 1, 2, \dots, n; k = 1, 2, \dots, N), \quad (4-12)$$

where

λ_{ij} - i^{th} undetermined Lagrangian multiplier at j^{th} time interval,

F_{ik}^* - i^{th} generalized equivalent nonpotential force, at k^{th} time interval,

$$\text{and} \quad S_1 = \sum_{k=1}^N L_k^*(\varphi_{1k}, \varphi_{2k}, \dots, \varphi_{nk}, \dot{\varphi}_{1k}, \dot{\varphi}_{2k}, \dots, \dot{\varphi}_{nk}) \Delta t. \quad (4-13)$$

The 3nN first-order difference equations resulting from Equations (4-10) - (4-12) in conjunction with an appropriate difference expression for $\Delta \varphi_{ik}$ are obtained from only one differentiation and are in a form which allows their solution to be marched out with time. Furthermore, the differentiations $(\partial S_1 / \partial \varphi_i)$ and $(\partial S_1 / \partial \dot{\varphi}_i)$ produce the $_{ij}P_r^*$ terms and uncouple the equations in one step.

Example

In order to compare the two methods, the above procedure will be applied to the system in Figure 4-1. Since the internal spring $\bar{K}_{34}$ produces an internal torque which is transferrable to each input as a generalized equivalent torque (see Chapter II), it will be included in the F_{ik}^* terms, leaving the sum S_1 made up of only the kinetic energy of the system. Therefore,

$$S_1 = \frac{1}{2} \left\{ I_{11}^* \dot{\varphi}_1^2 + 2I_{12}^* \dot{\varphi}_1 \dot{\varphi}_2 + I_{22}^* \dot{\varphi}_2^2 \right\}, \quad (4-14)$$

$$F_{1k}^* = - \left(C_{11}^* \dot{\varphi}_1 + C_{12}^* \dot{\varphi}_2 \right)_k + T_{s/1,k-1}^* + K_{b/1k}^* \Delta \varphi_{1k} + T_1, \quad (4-15)$$

$$F_{2k}^* = - \left(C_{21}^* \dot{\varphi}_1 + C_{22}^* \dot{\varphi}_2 \right)_k + T_{s/2,k-1}^* + K_{b/2k}^* \Delta \varphi_{2k} + T_2, \quad (4-16)$$

$$\Delta \varphi_{ik} = \varphi_{ik} - \varphi_{ik-1}, \quad (4-17)$$

where the $C_{ij,k}^*$ and $K_{b/ik}^*$ are defined in Table 3-1.

Substitution of Equation (4-17) into Equation (4-12) reduces Equations (4-10) and (4-11) to the following forms

$$\dot{\lambda}_{i,k+1} = \left(\frac{\partial S_1}{\partial \dot{\varphi}_i} \right)_k + (F_i^*)_k, \quad (4-18)$$

and

$$\lambda_{i,k} = \left(\frac{\partial S_1}{\partial \dot{\varphi}_i} \right)_k. \quad (4-19)$$

Let $\dot{\lambda}_{i,k+1}$ be given by the finite difference relation

$$\dot{\lambda}_{i,k+1} = (\lambda_{i,k+1} - \lambda_{i,k}) / \Delta t. \quad (4-20)$$

Then Equation (4-18) becomes

$$\lambda_{i,k+1} = \lambda_{i,k} + \left[\left(\frac{\partial S}{\partial \varphi_i} \right)_k + (F_i^*)_k \right] \Delta t. \quad (4-21)$$

Substituting Equations (4-14) - (4-17) into Equations (4-12), (4-19), and (4-21), and solving for $\dot{\varphi}_{i,k+1}$ yields the following set of six equations

$$\begin{aligned} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}_{k+1} &= \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}_k + \left\{ \frac{1}{2} \begin{bmatrix} 11^{P*}_1 & 12^{P*}_1 & 21^{P*}_1 & 22^{P*}_1 \\ 11^{P*}_2 & 12^{P*}_2 & 21^{P*}_2 & 22^{P*}_2 \end{bmatrix}_k \begin{bmatrix} \dot{\varphi}_1 & 0 \\ \dot{\varphi}_2 & 0 \\ 0 & \dot{\varphi}_1 \\ 0 & \dot{\varphi}_2 \end{bmatrix}_k \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}_k \right. \\ &+ \left. \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}_k - \begin{bmatrix} C_{11}^* & C_{12}^* \\ C_{21}^* & C_{22}^* \end{bmatrix}_k \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}_k + \begin{bmatrix} T_{s/1} \\ T_{s/2} \end{bmatrix}_{k-1} + \begin{bmatrix} K_{b/1}^* & \Delta \varphi_1 \\ K_{b/2}^* & \Delta \varphi_2 \end{bmatrix}_k \right\} \Delta t, \quad (4-22) \end{aligned}$$

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}_{k+1} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}_k + \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}_k \Delta t, \quad (4-23)$$

and

$$\begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}_{k+1} = \begin{bmatrix} I_{11}^* & I_{12}^* \\ I_{21}^* & I_{22}^* \end{bmatrix}_{k+1}^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}_{k+1} \quad (4-24)$$

The initial values of the λ 's are obtained by solving Equation (4-19), using the initial values for the φ 's and $\dot{\varphi}$'s. Hence

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}_0 = \begin{bmatrix} I_{11}^* & I_{12}^* \\ I_{21}^* & I_{22}^* \end{bmatrix}_0 \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}_0 \quad (4-25)$$

The matrix Equations (4-22) - (4-24) require one less numerical step of integration as pointed out in [30]. It is also pointed out in reference [30] that the λ 's are the momenta, usually called p_i 's, and that Hamilton's canonical equations are obtained directly without formerly deriving the generalized momenta. These equations may now be marched out with time to obtain the solution to the dynamical equations of motion. They require one less integration step as opposed to solving Equations (4-8) and (4-9) numerically. The matrix inversion is only necessary once per integration and hence any error is only integrated once as opposed to twice for Lagrange's equations.

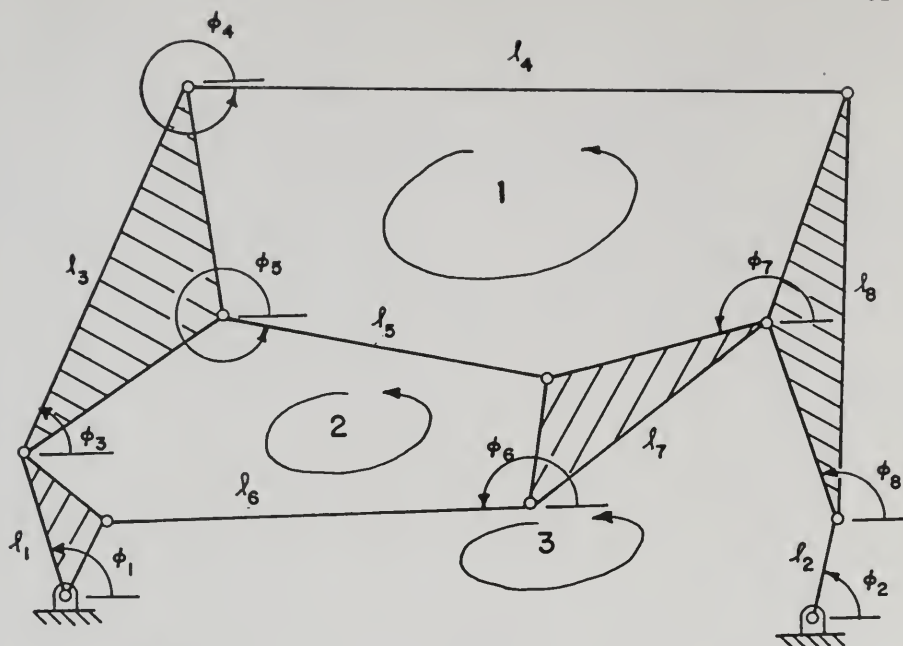
The significance of this development is the system formulation in terms of kinematic influence coefficients developed in Chapters II and III, allowing complex mechanisms with multiple inputs to be analyzed for their dynamic response by established numerical methods. This formulation provides a way to reduce complex, multiple input mechanisms to coupled equivalent mass systems, yielding differential equations of motion possessing variable coefficients. These coefficients are known in terms of the mechanism geometry through the kinematic influence coefficients of velocity and acceleration.

CHAPTER V

SUMMARY AND CONCLUSIONS

Complex multiple input linkage systems have been difficult to analyze for their dynamic response because of their nonlinear geometric character. This characteristic generates holonomic constraints associated with the generalized coordinates necessary in describing the motion of the linkage system. Large numbers of generalized coordinates (see Figure 5-1(a)) have been required to obtain the system's dynamical equations of motion. Algebraic equations of constraint are required to account for the generalized coordinates other than those associated with independent system inputs. The result is a large number of coupled, nonlinear, second-order differential equations together with a set of algebraic equations in terms of undetermined Lagrangian multipliers which account for the geometrical constraints on the system. The algorithm required to integrate numerically and solve this set of equations requires a dual iteration scheme, one to solve the differential equations of motion and one to satisfy the geometric constraints on the linkage system.

Methods such as those developed by Chace [11], Uicker [13], and Carson and Trummel [14], utilizing relative coordinates and 4×4 matrix coordinate transformations, have been the only tools available to solve the dynamic response question for these systems. The set of



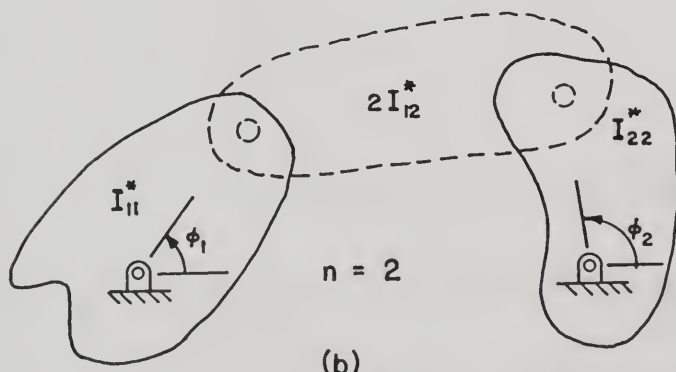
$$N = 8$$

$$n = 2$$

8 Generalized Coordinates

6 Algebraic Equations of Constraint

(a)



(b)

Figure 5-1. Complex Multiple Input Mechanism and Its Coupled Equivalent Mass System

second-order differential equations resulting from their methods has been sufficient, though unwieldy, time-consuming, and subject to error, to describe the dynamical behavior of linkage systems.

The goal of this dissertation has been threefold: to develop a systematic method whereby linkage systems of high-order-complexity can be constructed from Assur groups in terms of kinematic influence coefficients of velocity and acceleration of the basic system group; to reduce these highly complex linkage systems to coupled, equivalent mass systems acted upon by equivalent variable rate springs, variable coefficient viscous dampers, and equivalent external forces and torques; and, to determine the differential equations of motion for the coupled equivalent mass system in terms of the minimum number of generalized coordinates (i.e., the number of independent system inputs).

The construction of general linkage systems of higher-order-complexity, as discussed in Chapter II, is seen to be expressed in terms of series and parallel link connections. The connection types are defined by multiplication (series) and addition (parallel) of successive velocity influence coefficients. The use of Assur groups to construct mechanisms of higher-order-complexity from existing system groups, without modifying the mobility of the basic chains, allows the displacement analysis to be performed by established procedures. This is the basis for eliminating the holonomic constraints on the system, reducing the number of generalized coordinates required to describe the motion from N (number of system links) to n (number of independent system inputs).

Elimination of the holonomic constraints subsequently reduces complex, multiple input linkage systems (see Figure 5-1(a)) to coupled,

equivalent mass systems (Figure 5-1(b)), requiring only n generalized coordinates to describe its motion. This reduction eliminates the need for relative coordinates and their matrix transformations required by the existing methods. Rather, the coefficients of the resulting differential equations of motion become known variables of the system's independent input parameters expressed in terms of kinematic influence coefficients.

Second-order differential equations describing the dynamical behavior of the equivalent mass system have been derived by the classical Lagrangian method, while first-order difference equations were derived by the direct application of Hamilton's principle. This method [18], yields a set of first-order difference equations derived through only one differentiation. These difference equations can be marched out with time, requiring only one matrix inversion per integration step.

The equivalent mass system formulation developed here provides a convenient and unique medium through which many problems concerning the dynamical behavior of linkage systems can be studied and simulated.

The influence of elastic deformation of system links on the dynamic response of the linkage system primary input can be studied, based on a hinged beam model such as the one shown in Figure 5-2. The effect of bearing deformation on the input dynamic response could be investigated, based on the system models in Figure 5-3. The equivalent mass systems are shown below the system models.

The elimination of the holonomic constraints places complex linkage systems in a convenient form for directly applying the principles of optimal control theory. Optimal open loop control laws may

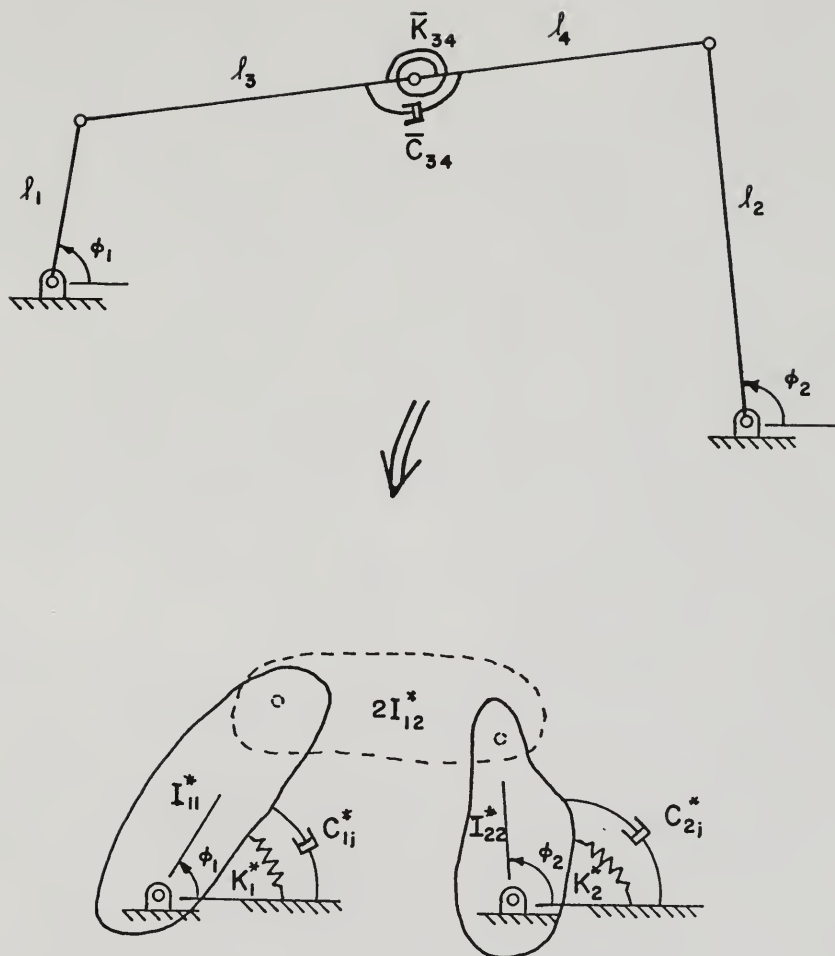


Figure 5-2. Linkage System with Elastic Coupler Link

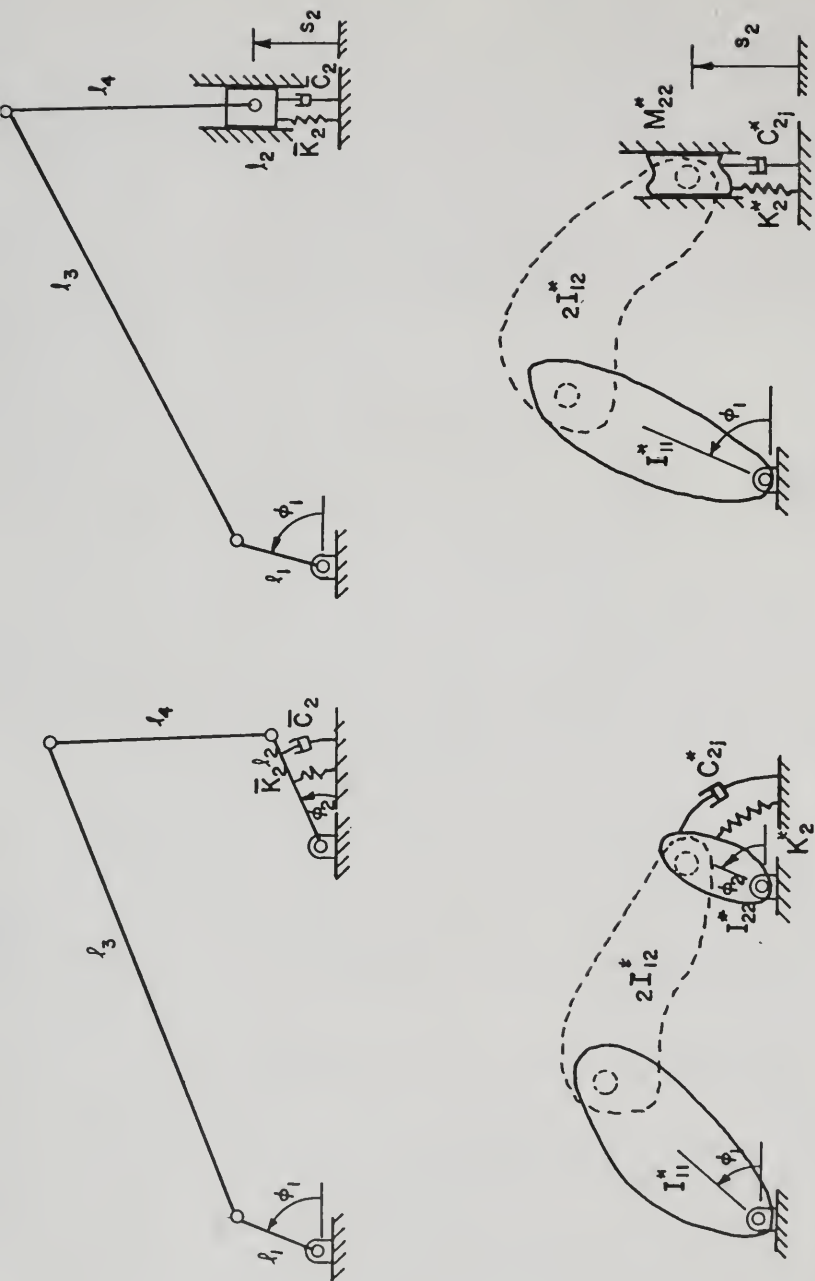


Figure 5-3. Linkage Models with Deformable Bearings

be determined which minimize specified performance indexes while satisfying prescribed constraints on the control itself, the states, or both. Figure 5-4 illustrates this concept as applied to complex linkage systems.

The problem could be formulated as follows: Determine the control u which minimizes the variation of the velocity $\dot{s}$ from the velocity $\dot{s}_1$ associated with position s_1 over the range $s_1 \leq s \leq s_2$, subject to the inequality constraint on the control, $\ddot{u} \leq \ddot{u}^*$. This in essence places an upper bound on the jerk $\ddot{u}$, thus giving third-order control for the cam surface designed to produce the required control u .

The preceding problems are not intended to be solved here. Rather, they are provided to point out possible research areas which can be pursued, utilizing the system formulation developed in this dissertation.

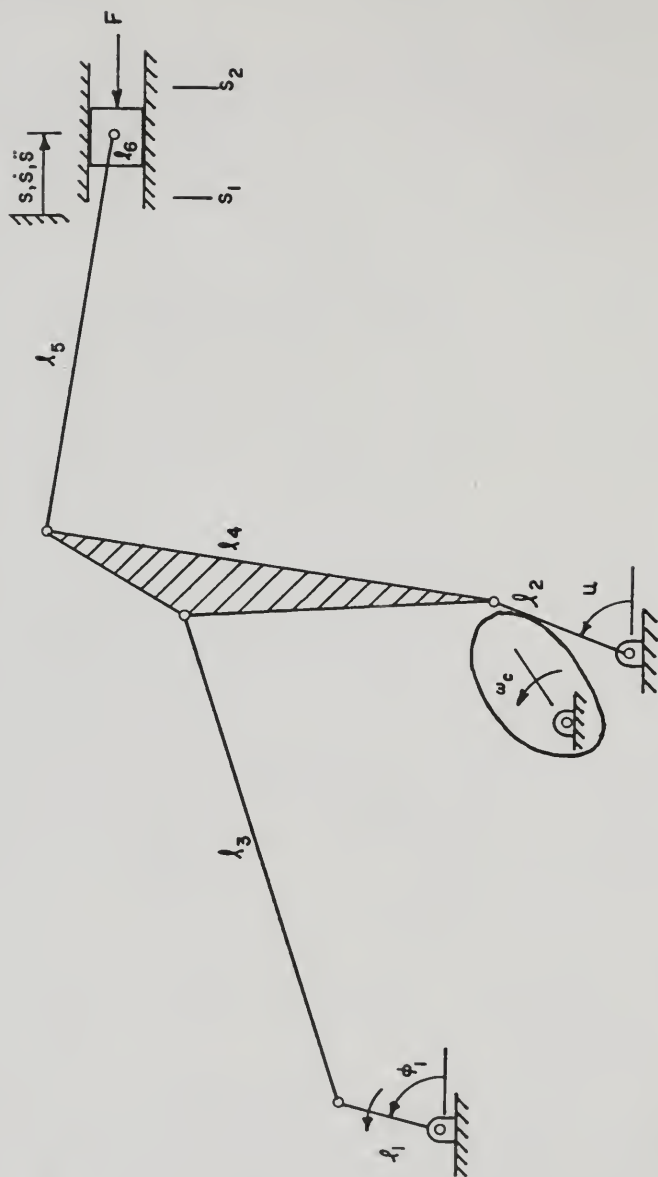


Figure 5-4. Optimal Open Loop Control Example

APPENDIX

APPENDIX A

DIRECT DERIVATION OF FIRST-ORDER DIFFERENCE EQUATIONS FOR DYNAMICAL SYSTEMS

The derivation of the set of first-order difference equations used in Chapter IV to solve the time response of complex linkage systems is presented in part from the paper [18] "Derivation of First-Order Difference Equations for Dynamical Systems by Direct Application of Hamilton's Principle" by Vance and Sitchin. The purpose for presenting this derivation is for completeness and convenience to the reader of this dissertation, since the method proves to be ideal for treating complex multiple input linkage systems. The nomenclature in this derivation does not correspond directly with that in the main text. It is therefore listed separately at the end of this appendix.

Derivation of Method

Hamilton's principle for nonconservative systems with k degrees of freedom is

$$\delta I = \delta \int_0^T L dt + \int_0^T \left(\sum_{i=1}^k F_i \delta q_i \right) dt = 0, \quad (A-1)$$

where the integrand of the second integral is the virtual work of the nonconservative forces. This second integral is zero for conservative systems. After partitioning the

interval 0 to T into N small increments ΔT , the two integrals can be approximated by sums and the principle can be rewritten as

$$\delta S_1 + S_2 = 0, \quad (A-2)$$

where the functions S_1 and S_2 are sums given by

$$S_1 = \sum_{n=1}^N L_n(q_{1n}, q_{2n}, \dots, q_{kn}, \dot{q}_{1n}, \dot{q}_{2n}, \dots, \dot{q}_{kn}) \Delta t, \quad (A-3)$$

$$S_2 = \sum_{n=1}^N \sum_{i=1}^k F_{in} \delta q_{in} \Delta t. \quad (A-4)$$

Equation (A-2) requires that the variation of the function S_1 equal the negative S_2 . Since the displacements and velocities are to be related by some finite-difference expression, they are not independent of each other. The problem lends itself to the use of Lagrangian multipliers in order to achieve an independent variation of coordinates. The equations of constraint, defined by the previously mentioned finite-difference relationship, have the general form

$$\dot{q}_{in} = \frac{\Delta q_{in}}{\Delta t}, \quad (A-5)$$

or

$$f_{in} = \Delta q_{in} - \dot{q}_{in} \Delta t = 0, \quad (A-6)$$

where Δq_{in} is any desired expression for the first-order difference of q_{in} .

Taking the variation of the constraint function (A-6)

gives

$$\delta f_{in} = \sum_{j=1}^N \left(\frac{\partial f_{in}}{\partial q_{ij}} \delta q_{ij} + \frac{\partial f_{in}}{\partial \dot{q}_{ij}} \delta \dot{q}_{ij} \right) = 0. \quad (A-7)$$

Equation (A-2) can now be written as

$$\delta S_1 + \sum_{n=1}^N \sum_{i=1}^k \lambda_{in} \delta f_{in} + S_2 = 0. \quad (A-8)$$

Substitution of Equation (A-7) into Equation (A-8) and rearrangement of terms gives

$$\begin{aligned} \sum_{n=1}^N \sum_{i=1}^k \left[\frac{\partial S_1}{\partial q_{in}} + F_{in} \Delta t + \sum_{j=1}^N \lambda_{ij} \frac{\partial f_{ij}}{\partial q_{in}} \right] \delta q_{in} \\ + \sum_{n=1}^N \sum_{i=1}^k \left[\frac{\partial S_1}{\partial \dot{q}_{in}} + \sum_{j=1}^N \lambda_{ij} \frac{\partial f_{ij}}{\partial \dot{q}_{in}} \right] \delta \dot{q}_{in} = 0. \end{aligned} \quad (A-9)$$

The kN λ_{ij} are chosen so that the kN bracketed expressions in the second double summation are all zero. This leaves the δq_{in} as independent variations. Equation (A-9) can then be satisfied by independently requiring the kN bracketed expressions in the first double summation to be zero. Thus the motion of the dynamical system will be such that the following $3kN$ equations are satisfied ($i = 1, 2, \dots, k$; $n = 1, 2, \dots, N$):

$$\frac{\partial S_1}{\partial q_{in}} + \sum_{j=1}^N \lambda_{ij} \frac{\partial f_{ij}}{\partial q_{in}} = -F_{in} \Delta t, \quad (A-10)$$

$$\frac{\partial S_1}{\partial q_{in}} + \sum_{j=1}^N \lambda_{ij} \frac{\partial f_{ij}}{\partial q_{in}} = 0, \quad (A-11)$$

$$f_{in} = \Delta q_{in} - \dot{q}_{in} \Delta t = 0. \quad (A-12)$$

When the first-order difference form Δq_{in} is substituted into Equation (A-12), the summations in Equations (A-10) and (A-11) are reduced to only a few terms. For example, if

$$\Delta q_{in} = q_{in} - q_{in-1},$$

$$\frac{\partial f_{ij}}{\partial q_{in}} = 0, \quad j \neq n, \quad n+1; \quad \text{and} \quad \frac{\partial f_{ij}}{\partial q_{in}} = 0, \quad \text{if } j \neq n. \quad (A-13)$$

As is characteristic of the Lagrangian multiplier method, the convenience of treating dependent variables as if they were independent has been gained at the expense of an added set of unknowns, the λ 's. Unlike many applications of Lagrangian multipliers, however, the λ 's are not in general constant.* In fact the λ 's represent momenta, . . ., and have the status of independent coordinates [3]**.

The set of Equations (A-12) may be considered trivial (although necessary) in the sense that they are simply the equations of constraint between the velocities and displacements.

* The λ 's will be constant only in the case of ignorable coordinates.

** Goldstein, H., Classical Mechanics, Addison-Wesley Publishing Co., Atlanta, 1965, p. 227.

The derivation just presented is not tied to any particular finite-difference form. A determination of the best form to use will depend on the application.

The symbols are defined as

δ = "variation of"

Δ = finite increment

$\dot{}$ = dot appearing directly above a variable designates derivative with respect to time

λ = undetermined Lagrangian multiplier constant

i = the corresponding generalized coordinate

n = the corresponding time interval

j = the corresponding time interval

t = time

F_{in} = generalized nonpotential forces

APPENDIX B

NUMERICAL SOLUTION TO A TWO DEGREES OF FREEDOM EXAMPLE

The purpose of this appendix is to illustrate the actual responses obtained from the dynamical equations of motion derived for the five-bar mechanism in Chapter IV (see Figure (B-1(a))).

The procedure for obtaining the variable coefficients to Equations (4-22) - (4-25), describing the dynamical behavior of the system in Figure B-1(a), is described below. The necessary coefficients are $K_{\ell/ik}^*$, $C_{ij,k}^*$, $T_{i,k}^*$, $I_{ij,k}^*$, and $ij^P_{r,k}^*$.

- (1) For the given initial position of the mechanism defined by φ_1 and φ_2 , calculate the kinematic position information pertaining to the other links $\bar{\varphi}_3$, $\bar{\varphi}_4$, etc., by using the equations in Figure B-2.
- (2) Substitute this position information into the appropriate blocks of Table 2-1 to determine the velocity and acceleration influence coefficients ℓ_{gi}^g and ℓ_{ij}^h . For example,

$$3\ell_{gi}^g = \frac{3\bar{\varphi}_{1,k+1} - 3\bar{\varphi}_{1,k-1}}{2\Delta\varphi_1} . \quad (B-1)$$

- (3) Once the ℓ_{gi}^g 's and ℓ_{ij}^h 's are determined, substitute them into the appropriate blocks of Tables 3-1 and 3-2. This furnishes the following expressions for the coefficients to the differential equations:

$$K_{l/ik}^* = \sigma_{l^E_{ik}} \bar{K}_l \left\{ \frac{l^E_{ik-1}{}^2 + l^E_{ik}{}^2}{2} \right\}, \quad (B-2)$$

$$C_{ij,k}^* = \sum_{l=1}^N \left(\bar{C}_l (l^E_i) (l^E_j) \right)_k, \quad (B-3)$$

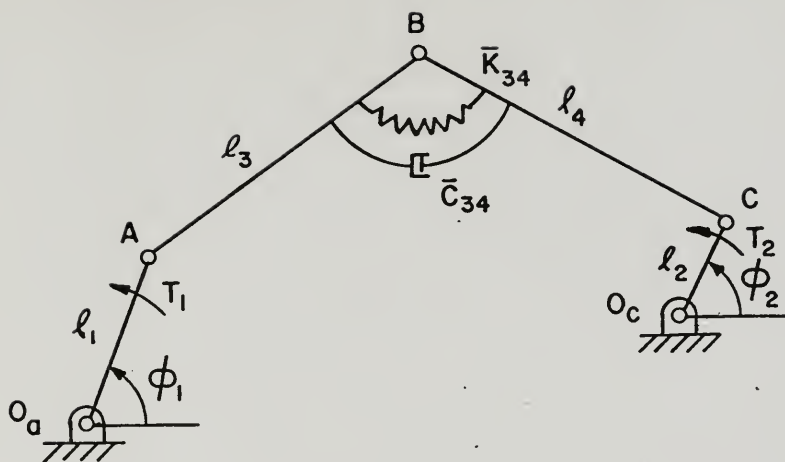
$$T_{ik}^* = (l^E_{ik}) \bar{T}_l. \quad (B-4)$$

$I_{ij,k}^*$ and $i_{j,r,k}^*$ are determined from the expressions in Table 3-2, blocks 1.1 and 1.2. The right-hand side of Equations (4-22) - (4-25) are now completely known, allowing the $k+1^{st}$ λ 's, φ 's and $\dot{\varphi}$'s, to be determined. This procedure is repeated until the equations of motion are integrated over a predetermined time interval. Equations (4-22) - (4-25) represent the equations of motion for the coupled equivalent mass system shown in Figure (B-1(b)).

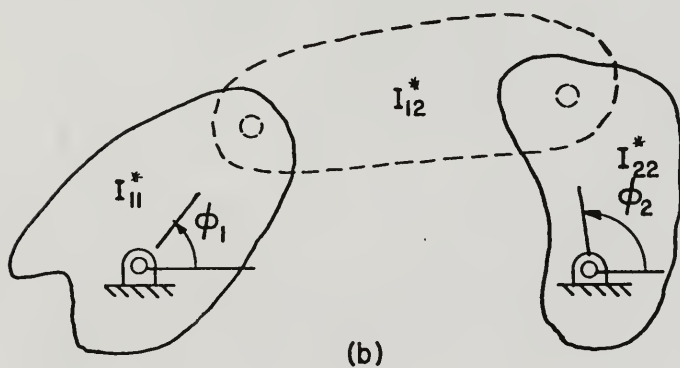
The solutions to the dynamical equations of motion are shown in Figure (B-3) and (B-4). Polar plots of $\dot{\varphi}_1$ vs φ_1 and $\dot{\varphi}_2$ vs φ_2 are shown for the parameters listed in Table B-1.

For added clarity, Figure (B-5) illustrates the equivalent inertias as functions of time. The equivalent inertias show no cyclic phenomenon due to the noncyclic character of the input links. However, the inertia coupling term, I_{12}^* , illustrates the coupling between the positive inertia terms I_{11}^* and I_{22}^* .

The influence coefficients were calculated by finite differences. The total problem was programmed on the IBM 360-65. The program consisted of 270 cards and required 0.38 minutes to execute for an integration step size of $\Delta t = .000025$ second. This step size can be increased considerably without affecting the solution accuracy, thus decreasing the execution time by that factor.

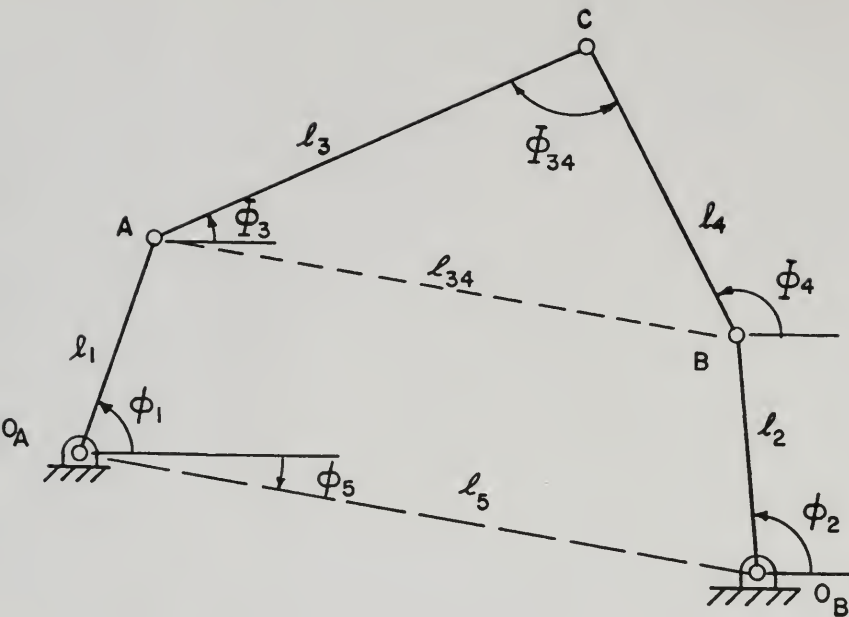


(a)



(b)

Figure B-1. Two Degrees of Freedom Five-Bar Example



$$l_{34}^2 = l_3^2 + l_4^2 - 2l_3l_4\cos\phi_{34}$$

$$l_{34}^2 = (l_5\cos\phi_5 - l_1\cos\phi_1 + l_2\cos\phi_2)^2 + (l_5\sin\phi_5 + l_1\sin\phi_1 - l_2\sin\phi_2)^2$$

$$\Phi_{34} = \cos^{-1} \left[\frac{l_3^2 + l_4^2 - l_{34}^2}{2l_3l_4} \right]$$

$$\Phi_3 = \sin^{-1} \left[\frac{l_5\sin\phi_5 + l_1\sin\phi_1 - l_2\sin\phi_2}{l_{34}} \right] + \sin^{-1} \left[\frac{l_4\sin\phi_{34}}{l_{34}} \right]$$

$$\Phi_4 = \sin^{-1} \left[\frac{l_5\sin\phi_5 + l_1\sin\phi_1 - l_2\sin\phi_2}{l_{34}} \right] + \sin^{-1} \left[\frac{l_3\sin\phi_{34}}{l_{34}} \right]$$

Figure B-2. Kinematic Position Equations

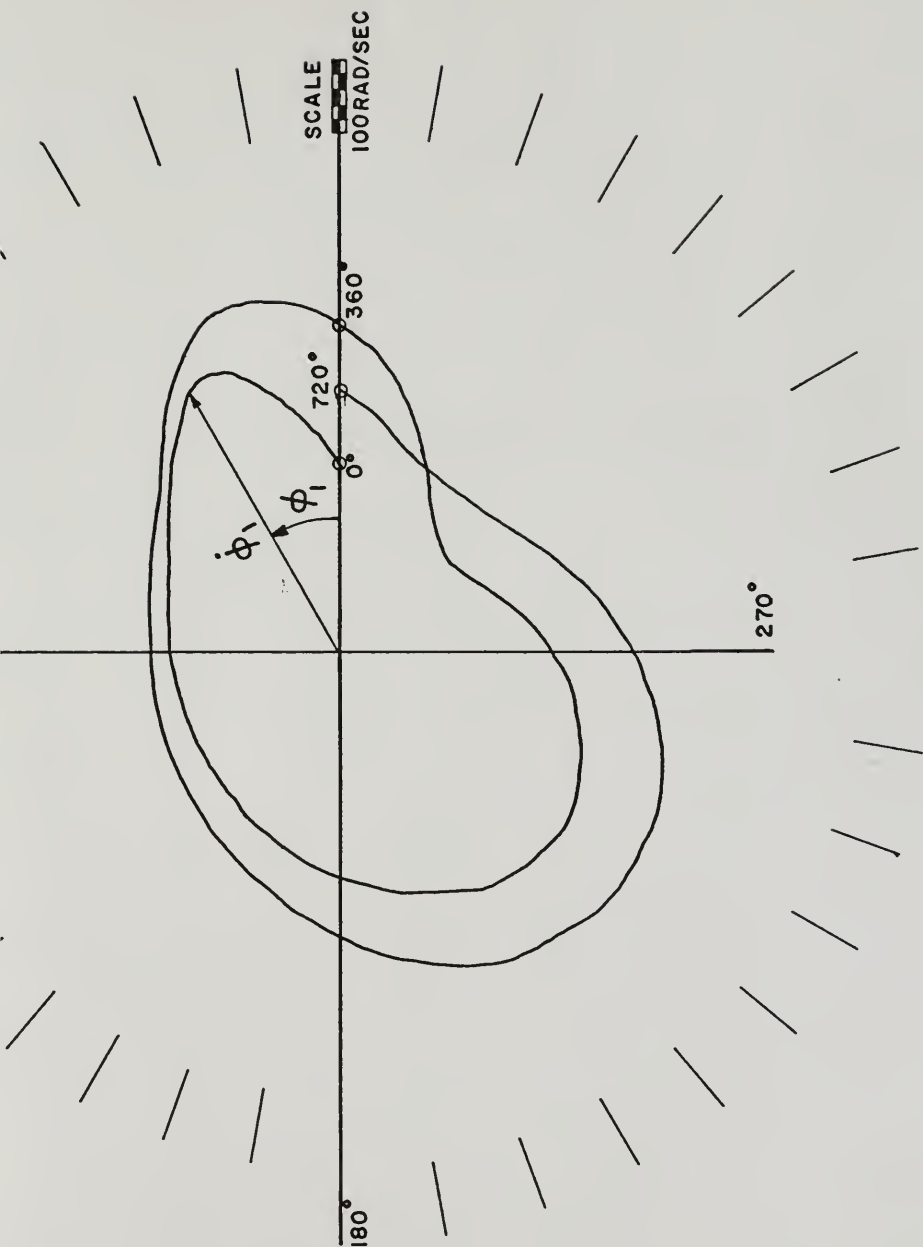


Figure B-3. Polar Phase Plane: $\dot{\phi}_1$ vs ϕ_1

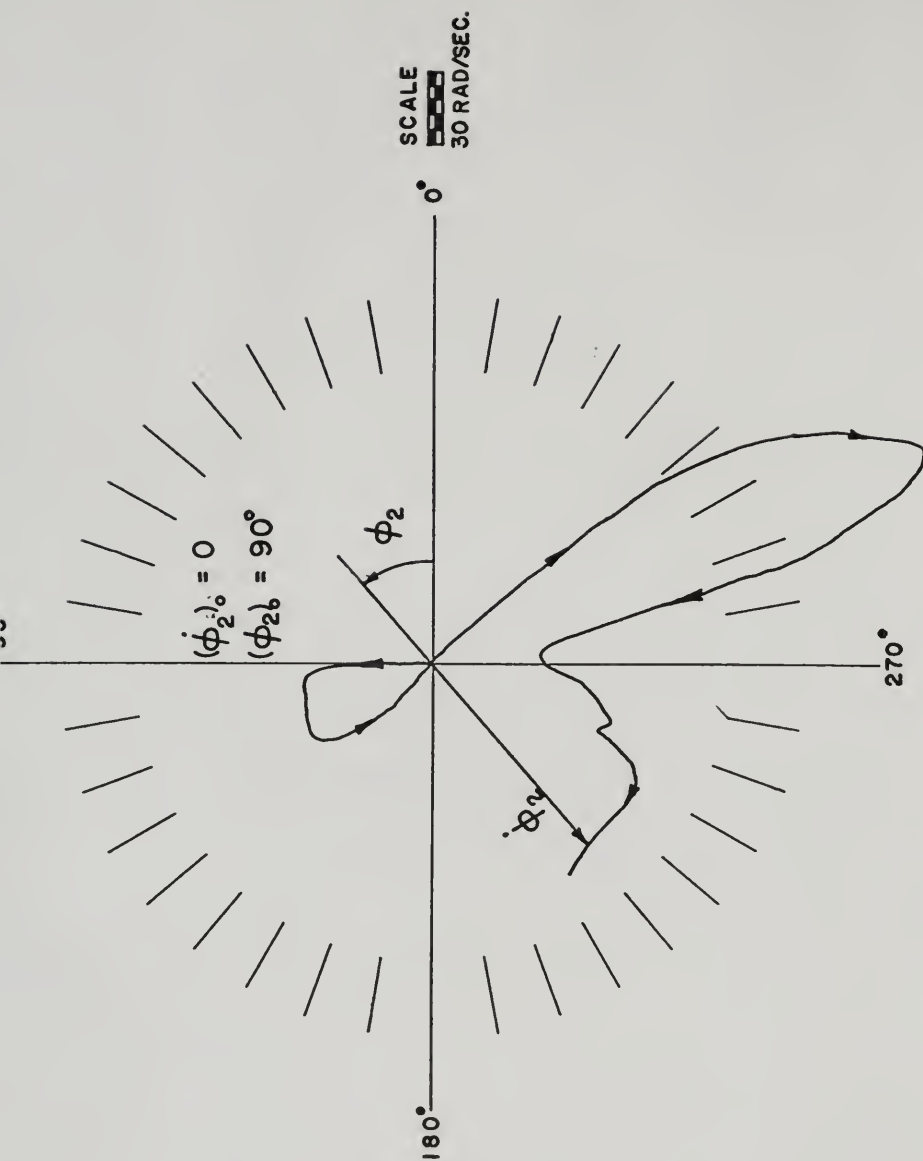


Figure B-4. Polar Phase Plane: $\dot{\phi}_2$ vs ϕ_2

TABLE B-1
FIVE-BAR PARAMETERS

	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5
Length (inches)	4.0	2.0	8.0	12.0	11.18
Weight (lbs.)	2.97	1.48	5.52	4.16	- 0 -
Moment of inertia (in. lb. sec. ²)	.0488	1.14	.1385	.4062	- 0 -

$$\bar{K}_{34} = 50.0 \text{ in. lb./rad.}$$

$$\bar{C}_{34} = 5.0 \text{ in. lb. sec./rad.}$$

$$T_1 = .05 \text{ in. lb.}$$

$$T_2 = -.02 \text{ in. lb.}$$

$$(\varphi_1)_0 = 0^\circ$$

$$(\dot{\varphi}_1)_0 = 250 \text{ rad./sec.}$$

$$(\varphi_2)_0 = 90^\circ$$

$$(\dot{\varphi}_2)_0 = 0$$

Solution to differential equations for known forcing functions, constant T_1 and T_2 , are shown on Figures (B-3) and (B-4).

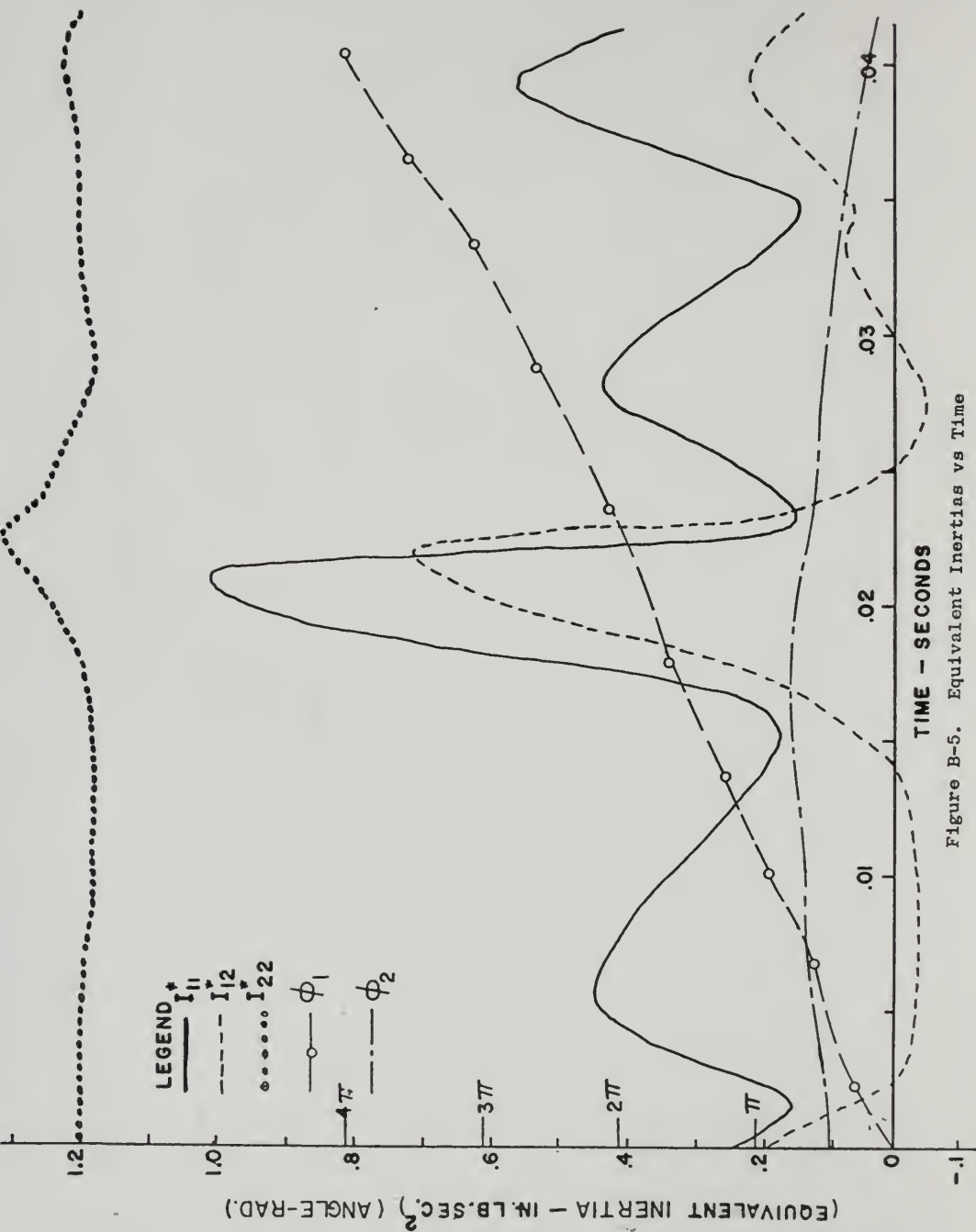


Figure B-5. Equivalent Inertias vs Time

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BIOGRAPHICAL SKETCH

Charles Edward Benedict was born March 21, 1939, at Tallahassee, Florida. He graduated from Leon High School in June, 1957. He began studies at Florida State University in 1958, and graduated with the degree Bachelor of Science with a major in Mathematics in December, 1963. After working for three and one-half years for Florida Gas Transmission Company, he began studies at the University of Florida in the field of Mechanical Engineering in April, 1967. He graduated with the degree Bachelor of Science in Engineering with high honors in August, 1968. He received an Engineering College Fellowship and continued his advanced education, receiving a Master of Science in Engineering from the University of Florida in December, 1969. He was awarded a NDEA Title IV Fellowship and continued his studies toward a degree of Doctor of Philosophy. This dissertation completes these studies.

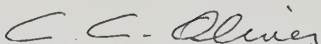
Charles Edward Benedict is married to the former Patricia Ann Casey and has one daughter, age seven. He is a member of Kappa Alpha Order, Tau Beta Pi, Pi Tau Sigma, Phi Kappa Phi, Florida Engineering Society, and the American Society of Mechanical Engineers.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



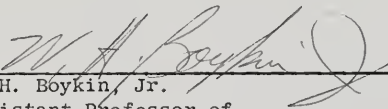
D. Tesar, Chairman
Professor of Mechanical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



C. C. Oliver, Co-Chairman
Professor of Mechanical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



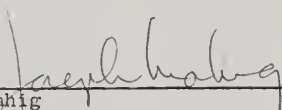
W. H. Boykin, Jr.
Assistant Professor of
Engineering Science & Mechanics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



T. E. Bullock
Associate Professor of Electrical Engineering

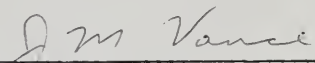
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J. Mahig

Associate Professor of Mechanical Engineering

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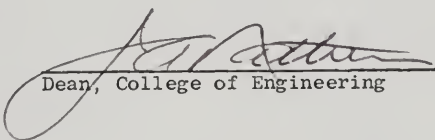


J. M. Vance

Assistant Professor of Mechanical Engineering

This dissertation was submitted to the Dean of the College of Engineering and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December, 1971



Dean, College of Engineering

Dean, Graduate School

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